PHYS 2170: General Physics 3 for Majors **Fall 2021**

Exam 3

- This exam is open book and open notes. You are not allowed to use any electronic devices besides a calculator.
- You have 2 hours to complete this exam (unless Disability Services has authorized a request for extended time: 50% extra time is 3 hours, 100% extra time is 4 hours). Good luck!

20 **Problem 1:** Let $A > 0$ be constant. Consider a quantum particle in one dimension with wave function

$$\Psi(x) = \begin{cases} A & 0 < x < L \\ 2A & L < x < 3L \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

1.1. Fix the constant A by demanding the wave function is normalized.

1.2. What is the probability that we measure x in the domain $2L < x < 3L$?

15 **Problem 2:** In a universe where the "electron" had spin-$\frac{3}{2}$ instead of spin-$\frac{1}{2}$, how many protons would you expect in the nucleus of the first two "noble gases"? In a noble gas atom, all allowed states at a fixed single particle energy level are occupied by the "electron". Assume that (besides spin) the allowed quantum states for particle in a Coulomb potential are unchanged. Be sure to justify your answer!

Problem 3: Consider a toy model for an "atom" which consists of two particles of equal mass m, spinning around each other in circular orbits, and connected by a "rope" of variable length ℓ but constant tension T. Assume the particles spin around their midpoint, and move in the xy-plane.

10 **3A:** Let's solve the classical dynamics of this system.

3A.1. Using Newton's Second Law, relate the velocity of each particle to ℓ, m and T.

3A.2. Show that the total energy of the system is[1] $E = \frac{3}{2}T\ell$.

3A.3. Write the (magnitude of the) angular momentum L_z in terms of energy E. Show that

$$L_z = \sqrt{\frac{4m}{27T^2}E^3}. \tag{2}$$

15 **3B:** Let's now estimate the quantum behavior of this system.

3B.1. Using Bohr's quantization argument for angular momentum L_z (not the exact quantization we later described), estimate the allowed energy levels. As in Bohr's model, assume no states have $L_z = 0$.

3B.2. If the angular momentum L_z can only change by $\pm\hbar$ when this "atom" emits or absorbs a photon, what photon wavelengths can be emitted/absorbed by this system?

[1] *Hint:* Be careful – there are two particles in motion here! What is the potential energy stored in the rope?

Problem 4: Consider a two-dimensional "infinite well" in which a quantum particle of mass m is constrained to lie in the domain $0 \le x \le L$ and $0 \le y \le 3L$. There is no potential energy in the well.

20 **4A:** Let us first find the allowed energy levels E in this well.

- 4A.1. Plug in ansatz $\Psi(x,y) = X(x)Y(y)$ into the time-independent Schrödinger equation relating Ψ and E. Use separation of variables to find ordinary differential equations for X and Y.
- 4A.2. Solve for the possible choices of $X(x)$ and $Y(y)$ with correct boundary conditions.
- 4A.3. What are the allowed values of energy levels E in the box?

5 **4B:** Find an energy level E which is degenerate.

Problem 5 (Buckyball): A buckyball is a large molecule consisting of carbon atoms arranged in a spherical pattern. Model an electron moving along the buckyball as a particle of mass m moving close to the surface of a sphere of radius a. a is related to the number of carbon atoms in the buckyball.

10 **5A:** Let us begin by assuming that the electron can be approximated by a particle moving *exactly* on the surface of a two dimensional sphere. Then the rotational energy levels are

$$E_{\text{rot}} = \frac{\mathbf{L}^2}{2ma^2}. \tag{3}$$

What are the allowed energy levels of E_{rot} in quantum mechanics?

15 **5B:** In reality, the electron bound to the buckyball can move a little bit in the radial direction. The total energy of the particle $E = E_{\text{rot}} + E_{\text{rad}}$, where E_{rad} is associated with the motion of the electron in the radial direction. A crude model is that, if b is the "size" of a single carbon atom,

$$E_{\text{rad}} = \frac{p^2}{2m} + \frac{E_*}{2}\left(\frac{x}{b}\right)^2, \tag{4}$$

where p is (radial) momentum and $x = r - a$ is the deviation in radial coordinate from $r = a$.

Use the Heisenberg uncertainty principle for x and p to estimate the lowest value of E_{rad}:

$$E_{\text{rad}} \approx \frac{\hbar}{2b}\sqrt{\frac{E_*}{m}}. \tag{5}$$

5 **5C:** We can estimate E_* using what we know about atomic physics.

- 5C.1. If E_* is related to the binding energy of the electron to a carbon atom, estimate how E_* should scale with a, b, $\hbar$ and m. (Think about what E_* would be if the atom was hydrogen!)
- 5C.2. Note that $b \ll a$. Would you expect the low energy motion of the electron to be primarily rotational, or in the radial direction?

10 **Problem 6 (Ionization):** Consider a toy model for an atom in which the electron of mass m and charge $-e$ lives in one dimension. The atom is represented by an unknown potential $U(x)$, which tends to 0 as $|x| \to \infty$, and which admits a local bound state where the electron has an energy of $E = -\varphi$ (here $\varphi > 0$ is positive). If a weak electric field of strength $\mathcal{E}$ is applied, then we observe that after waiting a typical time scale t_{ionize}, the electron is ripped off from the atom by the field. This process is called **ionization**.

Using the basic theory of quantum mechanics which we have developed, argue that for small $\mathcal{E}$,

$$t_{\text{ionize}} \sim \mathrm{e}^{\mathcal{E}_*/\mathcal{E}}, \tag{6}$$

and estimate $\mathcal{E}_*$ in terms of m, $\hbar$, e and φ. Don't worry about factors of 2, π, etc., in your answer; but, do carefully explain the $\mathcal{E}$-dependence in (6).