PHYS 2170: General Physics 3 for Majors **Fall 2021**

Homework 10

Due: November 18 at 11:59 PM. Submit on Canvas.

Problem 1: Consider a non-relativistic quantum particle of mass m in a one dimensional box $0 \le x \le L$.

20 **1A:** It is in the state with the n^{th} smallest energy.

1A.1. What is the n^{th} allowed quantum state, $\Psi_n(x)$? Make sure it is normalized!

1A.2. If the particle is in the n^{th} state, what is the probability $\mathbb{P}(\frac{1}{3}L \le x \le \frac{2}{3}L)$?[1]

5 **1B:** Discuss what happens as n becomes large. Argue your answer reproduces what you would expect in a classical limit, where an energetic particle is just bouncing between the walls of the box.

Problem 2 (Two dimensional electron gas): The electrons in a thin film of semiconductor, such as GaAs, can be approximately treated as a two-dimensional electron gas. They have an effective mass (in this material) of $m \approx 5 \times 10^{-32}$ kg, which is much smaller than the actual electron mass (why this happens is a topic for future classes!). Model the thin film of GaAs as having infinite extent in the x and y directions, while only extending from $0 \le z \le a$ in the third direction.

20 **2A:** Let us begin by solving the time-independent Schrödinger equation.

2A.1. Plug in the ansatz $\Psi(x,y,z) = X(x)Y(y)Z(z)$ into the time-independent Schrödinger equation, and reduce the problem to ordinary differential equations for X, Y and Z.

2A.2. Show that the allowed solutions are (up to normalization)

$$X(x) = \mathrm{e}^{\mathrm{i}k_x x}, \qquad Y(y) = \mathrm{e}^{\mathrm{i}k_y y}, \qquad Z(z) = \sin\frac{n\pi z}{a}, \tag{1}$$

where $-\infty < k_x, k_y < \infty$ and $n = 1, 2, 3, \ldots$.

2A.3. Deduce the allowed energy levels E.

10 **2B:** We can think of this dispersion relation as giving rise to a set of many two-dimensional models of electrons, labeled by the integer n.[2] However, in an experiment we cannot realistically see all of these particles – we can only see the ones whose energy E obeys (here T is temperature)

$$E - E_0 \lesssim k_{\mathrm{B}}T, \tag{2}$$

where $k_{\mathrm{B}} \approx 1.4 \times 10^{-23}$ J/K, and E_0 is the smallest possible allowed energy value (found above).

2B.1. Show that there is a critical temperature T_{c}, well below which we can only see a *single* type of "two dimensional particle" (one value of n).

2B.2. In a typical experiment we might find $a \sim 30$ nm. Below what temperature will the electrons in GaAs behave as if they are a single species of particles which lives in two dimensions?

[1] *Hint:* Use the identity $\sin^2 z = \frac{1}{2}(1 - \cos(2z))$ to evaluate an integral that you find.

[2] This sort of idea also arises in string theory, where the vibrations of "strings" in extra dimensions give rise to different particles when viewed in the "large" dimensions.

25 **Problem 3 (Scanning tunneling microscope):** A **scanning tunneling microscope** works by taking a thin tip of metal and placing it *extremely* close to the surface of another metal. We can model the dynamics of an electron moving along the tip as follows:

$$U(x) = \begin{cases} 0 & 0 \le x \\ \varphi & 0 < x < d \\ 0 & x > d \end{cases}, \tag{3}$$

where d denotes the distance from the tip to the metal, and φ represents a work function for the metals (we assume for simplicity they are identical, and have chosen to "zero" our potential energy so that electrons moving in the metal have no potential energy).

We try to push a current through the metallic tip such that the electrons *tunnel* into the metal. In reality, this is achieved by having a voltage difference V between the tip and the surface. By Ohm's Law, we expect a current

$$I = \frac{V}{R} \tag{4}$$

will arise. Assume that the resistance of tunneling obeys

$$R = \frac{R_0}{P}, \tag{5}$$

where P denotes the probability that an incident electron at the tip will tunnel through the tip into the metal. Assume that typical electrons in the device have a kinetic energy of magnitude $K < \varphi$.

3.1. Using results from Lecture 32, argue that the probability

$$P = \mathrm{e}^{-\kappa d}, \tag{6}$$

and give a formula for κ in terms of m, K, φ and $\hbar$.

3.2. Assume that $m \approx 10^{-30}$ kg, $\varphi \approx 2$ eV, and $K \approx 0.05$ eV. What is the numerical value of κ?

3.3. A typical atom (in a metal) has a size of around 0.1 nm. If there were a defect on the surface of the metal – namely, an extra atom sitting on top of the crystal lattice, it could reduce the distance from the tip to the surface (when the tip hovers over the defect). Should it be possible to detect this defect by measuring a difference in tunneling current I?

20 **Problem 4:** Suppose that you have a particle of mass m in a two-dimensional square-shaped box (with hard walls), confined to the domain $0 \le x, y \le L$. Suppose that the probability that the particle will be found in the vicinity of the point

$$(x_0, y_0) = \left(\frac{L}{2}, \frac{L}{3}\right) \tag{7}$$

is vanishingly small, namely $|\Psi(x_0, y_0)|^2 = 0$.

4.1. Write down all possible allowed wave functions (of definite energy).[3]

4.2. Which of the wave functions you wrote down above obey (7)?

4.3. What is the lowest possible energy of the state which satisfies (7)? Express your answer in terms of $\hbar$, L and m.

[3] *Hint:* Not much algebra is needed; borrow results from Lecture 33, and/or your solution to Problem 2.

Problem 5 (Quantum particle on a torus): The (2d) **torus** is a two-dimensional shape with the topology of a donut (as you may know from pop math), but actually the better way to think about the torus is that it is a two-dimensional plane with points periodically identified.

10 **5A:** Let's warm up by thinking about a simpler example. You are surely familiar with the circle – in mathematics, this is the one-dimensional torus, and comes about from taking the real line $-\infty < x < \infty$ and just identifying the point x with the point $x + L$ for some constant L, which will correspond to the circumference of the circle. We can denote this mathematically as

$$x \sim x + L. \tag{8}$$

You can intuitively think of taking a number line and "gluing it" to itself (infinitely many times) in such a way that the points $0, L, 2L, \ldots$ all lie on top of each other after the gluing. Do this with a physical number line on e.g. a piece of string, and you'll have formed a loop of string.

In quantum mechanics, it is really useful to think about a circle in this way because it makes finding the wave functions of particles on a circle of circumference L very easy: we simply solve the Schrödinger equation on the line and restrict to the wave functions $\Psi(x)$ which obey

$$\Psi(x) = \Psi(x + L). \tag{9}$$

for all x. (Recall we did something almost identical, mathematically, on Homework 6.). This means that $\Psi(x)$ would be a single-valued (i.e. well-defined) quantum wave function even on the circle.

For convenience in this whole problem, go ahead and "choose units" where you can set $\hbar = 1$ and the particle mass $m = 2\pi^2$. This funny choice will reduce clutter in the equations and make the algebra a bit tidier.

5A.1. Review/present the solution to the free particle Schrödinger equation on the line. What are the allowed solutions $\Psi(x)$ associated to a given energy E?

5A.2. Which of these energies E correspond to a solution which is compatible with (9)?

5A.3. Deduce that there is a discrete spectrum of possible energy levels for a quantum particle on a circle of circumference L:

$$E = \left(\frac{n}{L}\right)^2, \qquad (n = 0, 1, 2, \ldots). \tag{10}$$

10 **5B:** Let's now extend this to the (two-dimensional) torus. We can generate any torus by tiling the two-dimensional plane with rectangles – or more generally rhombi. Using the notation from above,

$$(x, y) \sim (x + a_x, y + a_y) \sim (x + b_x, y + b_y). \tag{11}$$

For the torus to be a sensible (finite area) shape, we need to make sure that the vectors (a_x, a_y) and (b_x, b_y) are not proportional (i.e. they are linearly independent, in the language of linear algebra).

5B.1. As before, begin by solving the Schrödinger equation in the two-dimensional plane.

5B.2. Generalize the criterion (9) to incorporate the requirement that $\Psi(x, y)$ is well-defined on the torus defined by (11). Conclude that the allowed energy levels E are characterized by two integers – and of course the vectors (a_x, a_y) and (b_x, b_y).

5 **5C:** Some of the tilings of the plane found above don't really represent physically different tori, but rather are rotated or re-scaled versions of the same torus. We might expect each of these transformations to do something simple to the energies we found before.

5C.1. Suppose that we re-scale the torus, so that (here $\lambda \neq 0$ is a real number)

$$(a_x, a_y) \to (\lambda a_x, \lambda a_y), \tag{12a}$$
$$(b_x, b_y) \to (\lambda b_x, \lambda b_y). \tag{12b}$$

Show that the allowed energies on the re-scaled torus are related in a very simple way to those on the original torus.

5C.2. Suppose that we rotate the torus so that

$$(a_x, a_y) \to (a_x \cos\phi - a_y \sin\phi, a_x \sin\phi + a_y \cos\phi), \tag{13a}$$
$$(b_x, b_y) \to (b_x \cos\phi - b_y \sin\phi, b_x \sin\phi + b_y \cos\phi). \tag{13b}$$

Show that the allowed energies E are unchanged.

5 **5D:** Using the re-scalings and rotations above, we can always choose

$$(a_x, a_y) = (1, 0). \tag{14}$$

Yet even upon fixing this choice, there are *still* an infinite number of equivalent choices of (b_x, b_y) which describe "the same torus". These choices are generated by applying the operations

$$(b_x, b_y) \to (b_x + 1, b_y), \tag{15a}$$
$$(b_x, b_y) \to \left(\frac{-b_x}{b_x^2 + b_y^2}, \frac{b_y}{b_x^2 + b_y^2} \right). \tag{15b}$$

These are called **modular transformations**. Loosely speaking, the former corresponds to just re-defining which points in the plane are treated as "fundamental", while the latter is a clever rescaling/rotation. You do not need to justify these facts.

5D.1. Describe how (15a) transforms the allowed energies E and comment.

5D.2. Describe how (15b) transforms the allowed energies E and comment.

The physical properties of a system on a torus should be (up to suitable re-scalings) invariant under modular transformations. This actually is a very strong constraint because mathematicians have categorized the possible ways that this can be achieved! This idea has proved useful in, e.g., string theory.