Homework 3

Due: September 16 at 11:59 PM. Submit on Canvas.

25 **Problem 1 (Linear accelerator):** One of the first ways that particle physics was done is via linear accelerators (e.g. SLAC at Stanford). These roughly operate in a simple manner: suppose that particles are constrained to move along the x-axis. At point $x = 0$, there is an electric potential of V, while at points $x \geq D$ there is an electric potential of 0. A positive charge $+q$, with mass m, begins at rest at $x = 0$.

1.1. What are the kinetic and potential energies of the particle at $x = 0$?

1.2. Using energy conservation, find a formula for the speed v of the particle once it reaches $x \geq D$, in terms of c, m, q, V and D.

1.3. At SLAC, we estimate that $D = 3000$ m, $q = 1.6 \times 10^{-19}$ C, $V \approx 5 \times 10^{10}$ V, and $m \approx 10^{-30}$ kg. Estimate the final velocity v. (It might be easier to calculate $c - v$!).

1.4. Once the charge is accelerated to its final velocity, how long does it see the SLAC accelerator to be (from its own frame)?

Problem 2: An observer in frame S views two balls, one at event A, $(ct, x) = (ct_0, -D)$, moving at velocity $v = c\beta$, and one at event B, $(ct, x) = (ct_0, D)$, moving at velocity $v = -c\beta$. Assume $D > 0$ and $\beta > 0$.

10 **2A:** Eventually, the observer in frame S will see the balls collide with each other. Suppose the collision (event C) happens at $(ct, x)_S = (0, 0)$.

2A.1. What is t_0? (Note that you should expect $t_0 < 0$.)

2A.2. One natural definition of the relative velocity v_{rel} of the balls is

$$v_{\text{rel}} = \frac{2D}{|t_0|}. \tag{1}$$

Explain why it is possible for v_{rel} to be larger than c, and why this does not violate the postulates of relativity.

20 **2B:** Now, let's study this collision in frame S', which moves (in the x-direction) at velocity $c\beta$ relative to frame S.

2B.1. Find the coordinates of the events A, B, C, in S'.

2B.2. By comparing where the left-moving ball is in events B and C, determine the velocity of the left-moving ball as seen in frame S'. Confirm it is smaller than c.

2B.3. Compare your answer above to the relativistic velocity addition formula. Should they agree? Why?

25 **Problem 3:** Suppose that Earth is under attack by an alien spaceship, which is traveling at velocity v towards the Earth. In the reference frame of the alien ship, every T seconds, it fires a shell of mass m towards the Earth at velocity u.

3.1. In the frame of the alien spaceship, suppose that the alien ship is at $x = 0$, and it fires one shell at $ct = 0$ and one shell at time $ct = T$. Describe the worldlines of each of the two shells.

3.2. Suppose that in the ship's frame, the Earth was at $x = D$ at time $t = 0$. (Suppose D is relatively large.) Write down an equation for the worldline of the Earth, in the reference frame of the ship.

3.3. Let A/B denote events when the first/second shell hits Earth. Find the coordinates of A and B in the ship's frame.

3.4. Use Lorentz transformations to find the coordinates of A and B in Earth's frame.

3.5. What is the time between shell strikes as viewed on Earth?

20 **Problem 4:** Consider the 4-velocity (U_t, U_x, U_y, U_z) as defined in Lecture 6.

4.1. By starting from the definition of proper time, together with (e.g.) $U_x = \mathrm{d}x/\mathrm{d}\tau$, show that

$$-c^2 = -U_t^2 + U_x^2 + U_y^2 + U_z^2. \tag{2}$$

4.2. Using (2), show that if a particle travels at a regular velocity $\mathbf{u}$ of magnitude less than c in one inertial frame, it will be less than c in all inertial frames.

Problem 5 (Relativity and electromagnetism): Historically, a critical discovery was that Maxwell's equations of electrodynamics are – miraculously – consistent *as is* with Einstein's theory of relativity.

A proof of this is provided in more advanced physics classes. In a nutshell, the full set of Einstein-Maxwell equations predicts waves propagating at the speed of light, and if these equations are carefully Lorentz transformed, one finds that this prediction remains. In this problem, we explore how the mixing of space and time within relativity leads to subtle effects in electromagnetism. Consider a wire with charge per unit length λ, oriented in the x-direction as shown in Figure 1. Think about the wire as an array of point charges q, separated by distance ℓ, such that $\lambda = q/\ell$. Suppose that in reference frame R, the wire is static, as are the charges, while reference frame S moves at velocity $\mathbf{v} = c\beta\hat{\mathbf{x}}$ relative to frame R.

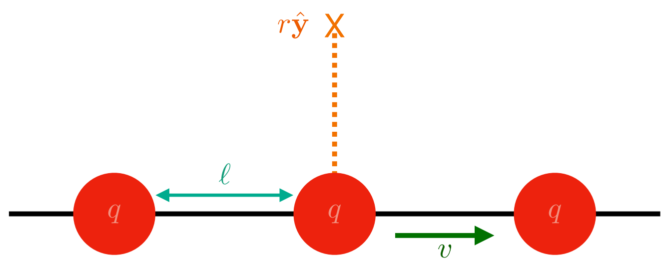

Figure 1: A wire of charge per unit length $\lambda = q/\ell$ in frame R. Frame S moves at velocity v parallel to the wire, and electric/magnetic fields are calculated at $r\hat{\mathbf{y}}$.

10 **5A:** Let's begin by using special relativity to calculate the charge density per unit length λ_S, along with the current I_S flowing (in the positive x-direction) down the wire, as measured in reference frame S.

5A.1. In frame S, the distance between point charges in the array will differ from what is observed in frame R, due to the relativistic nature of time and space. More quantitatively, show that the charge density per unit length in S is

$$\lambda_S = \gamma\lambda, \tag{3}$$

where $\gamma = (1 - \beta^2)^{-1/2}$ is our usual relativistic prefactor.

5A.2. To calculate the current I_S, imagine an observer at a fixed point in frame S watching the point charges move by. If the time interval between point charges passing by is τ (as viewed in frame S), then the current which the observer records is $I = q/\tau$. Show that

$$I_S = -\gamma v \lambda. \tag{4}$$

15 **5B:** Now let's try to generalize this calculation and explore some consequences of it.

5B.1. Deduce the form of the general Lorentz transformation between λ and I, when the relative velocity vector between frames is (as previously) aligned with the wire. You don't need a rigorous proof, but justify the answer as best you can.

5B.2. Use the laws of electromagnetism as you have previously learned them (a priori, without relativity) to calculate the electric and magnetic fields at the point $r\hat{\mathbf{y}}$, in a frame where there is a fixed λ and I. You should find that the only non-vanishing components of these fields are E_y and B_z.

5B.3. Deduce the Lorentz transformations betweeen E_y and B_z. Make sure to account for any possible effects coming from spacetime distortions and/or 5B.1. Note that

$$c^2 = \frac{1}{\mu_0 \epsilon_0}. \tag{5}$$

5B.4. Is it possible to find one inertial frame R in which there is an electric field but no magnetic field, and then another inertial frame S in which there is a magnetic field but no electric field? Explain your answer.