**PHYS 2170: General Physics 3 for Majors** **Fall 2021**

# Homework 4

**Due:** September 23 at 11:59 PM. Submit on Canvas.

**Problem 1 (Solar sail):** One proposal for how to achieve interstellar travel is to use a "solar sail". Analogous to how a sailboat uses a giant cloth sail to generate mechanical forces due to the wind, a solar sail would rely on forces due to pressure from electromagnetic radiaton – i.e. light from the sun.

15 **1A:** Let's first estimate the mechanical forces on our solar sail from our sun.

1A.1. Consider a photon of energy $E$ which approaches our solar sail at rest, and "head on". If the photon is reflected at 180° and has energy $E$ when reflected, what is the momentum that is imparted into the solar sail?[1]

1A.2. Suppose that the intensity of solar radiation on the solar sail is $I$. Intensity has units of power per unit area – namely $\mathrm{W/m^2}$ in SI units. If each photon carries exactly energy $E$, and travels in the same direction, show that the number of photons hitting an object of (perpendicular) cross-sectional area $A$ in time $\tau$ is

$$N = A\tau\frac{I}{E}. \tag{1}$$

1A.3. By using Newton's Second Law that $F = \mathrm{d}p/\mathrm{d}t$, along with Newton's Third Law, conclude that there is a force on a solar sail of area $A$ of magnitude:

$$F = \frac{2IA}{c}. \tag{2}$$

15 **1B:** Now, let's use Newton's Second Law to estimate how rapidly a solar sail can get our spaceship moving. Approximate that the solar sail moves at velocities $v \ll c$ (which you will justify later).

1B.1. Suppose the spaceship starts at rest. What is the acceleration of the ship? After applying (2), you may use non-relativistic mechanics for the remainder of the problem.

1B.2. If we want to travel a distance $D$ at constant acceleration $a$, how long does it take?

1B.3. What is our instantaneous velocity when we travel a distance $D$?

1B.4. Suppose that the spaceship has mass $m \approx 10^5$ kg (it can carry a few humans, food, supplies etc.). The intensity of solar radiation in outer space is $I \approx 1000\ \mathrm{W/m^2}$. Suppose that the solar sail is pretty large, and is a square of side length $L = 100$ m. We want to make it to Alpha Centauri, which is $\sim 10^{16}$ m away. How long will the journey take? Is this a reasonable strategy for human interstellar travel?

1B.5. Justify the use of non-relativistic mechanics in this calculation.[2]

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[1] Since a photon's energy $E$ is extremely small compared to the rest energy of the solar sail, we can neglect energy imparted into the sail.

[2] *Hint:* Does the ship reach speeds comparable to $c$?

**Problem 2 (Nuclear chain reaction):** A toy model for a (runaway) nuclear reaction is as follows: particle X is stable, but has an excited state X′, accessible when a neutron collides with X, which is unstable and decays into Y along with 2 neutrons:

$$\mathrm{X} + \mathrm{n} \rightarrow \mathrm{X}' \rightarrow \mathrm{Y} + \mathrm{n} + \mathrm{n}.$$

Let $m_{\mathrm{X}}$ denote the mass of X; use similar subscript notation for other masses.

15 **2A:** Let's first begin by thinking about this decay process.

2A.1. Explain why we need

$$m_{\mathrm{X}'} \geq 2m_{\mathrm{n}} + m_{\mathrm{Y}}. \tag{3}$$

2A.2. If the X′ particle moves very fast, it should have enough total energy to decay into the desired particles even if the X′ mass does not obey (3). Explain why, even if the X′ does move very fast, the decay process requires (3).

15 **2B:** Assume the X′ decay process is possible. Argue that if the neutrons carry away the most energy possible (in X′'s rest frame), then each neutron (in this frame) will be traveling at speed

$$v = c\sqrt{1 - \left(\frac{2m_{\mathrm{n}}}{m_{\mathrm{X}'} - m_{\mathrm{Y}}}\right)^2}. \tag{4}$$

15 **2C:** For this to drive a nuclear chain reaction, we now need to analyze the reaction where X absorbs a neutron. It would be desirable if the energetic neutron from the decay of X′ is energetic enough to push X into its excited state.

2C.1. In terms of the masses of the relevant particles, and $c$, find the critical velocity $v_*$ at which it is possible for a neutron to collide with an X at rest, and create an X′.

2C.2. Given a lump of matter full of X nuclei, give a qualitative argument why a "nuclear chain reaction" is possible if $v_* < v$.[3]

In reality, our model of the X′ state as difficult to create (requiring an energetic neutron) makes it undesirable for an actual nuclear reactor – one typically works with more unstable nuclei that can absorb low energy neutrons. Nevertheless, you should get a qualitative picture for how energy and mass get converted in nuclear reactions.

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[3] *Hint:* A nuclear chain reaction will require some ample supply of both X and neutrons, with enough energy to create an excited state X′. If we start with a single energetic neutron, should we expect to have more or less after the first collision?

25 **Problem 3 (Ultra high energy cosmic rays):** There is a theoretical upper limit called the **Greisen-Zatsepin-Kuzmin limit** on the energy of particles in cosmic rays. This upper limit can be estimated as follows. Firstly, we assume that the cosmic rays consist of protons of mass $m \approx 900$ MeV/$c^2$ – a theoretical assumption not always born out in practice, but reasonable for this problem. Secondly, in the Earth's reference frame $S$, the universe is filled with cosmic microwave background (CMB) radiation: photons of typical energy $E_{\text{CMB}} \approx 2 \times 10^{-10}$ MeV, moving both left and right in the $x$-direction (we do not need to consider the $y$ and $z$ directions in this problem). Thirdly, we assume the following process is allowed by particle physics: $\mathrm{p} + \gamma \to \mathrm{p} + \pi + \pi$, where $\pi$ denotes a pion of (approximate) mass $\frac{1}{6}m$.[4]

Suppose that we have a very fast moving proton, traveling in the $x$-direction at speed $v = \beta c$. Let $S'$ denote a reference frame in which this proton is at rest.

3.1. Using Lorentz transformations on the energy and momentum of a photon, estimate the largest possible energy $E_{\max}$ of the CMB photons in frame $S'$; express your answer in terms of $\beta$ and $E_{\text{CMB}}$.

3.2. In frame $S'$, consider the scattering process where the proton absorbs a photon. Explain why it is not possible to create pions unless $E_{\max} > \frac{1}{3}mc^2$.

3.3. Argue that it would be possible to create pions if $E_{\max} = \frac{7}{18}mc^2$.[5]

3.4. If the proton is moving so fast that it sees CMB photons of energy $E_{\max}$, it will generally be slowed down by collisions with CMB photons, until the CMB photons have energies below $E_{\max}$ in the proton's rest frame. Using your answers above, and the numbers from the problem description, estimate the maximal energy of a proton in a cosmic ray.

15 **Problem 4 (Accelerating particle):** Suppose we have a particle of mass $m$ and charge $q$, placed in a uniform electric field of strength $E$. Taking into account relativity, what will be the dynamics of the particle? It turns out that $F = qE$ still holds in relativity: namely, this particle will experience a constant force.

4.1. Solve for the trajectory of the particle, assuming that it starts from rest at $x(t = 0) = 0$. Show that

$$x(t) = \frac{mc^2}{qE}\left[\sqrt{1 + \left(\frac{qEt}{mc}\right)^2} - 1\right]. \tag{5}$$

4.2. Check that at small time $t$, your answer agrees with non-relativistic mechanics.

4.3. Show that there is a time $t = t_0$ such that, if we shoot a photon at the particle at time $t > t_0$ from $x = 0$, the photon will never reach the particle.

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[4] This is a simplification of the actual processes that go on, but is a good approximation for this problem.

[5] *Hint:* Assume that after the collision, the proton and 2 pions moved together at a constant velocity, as if they were one particle of mass $\frac{4}{3}m$. You should find that such a collision is possible at this value of $E_{\max}$.