PHYS 2170: General Physics 3 for Majors **Fall 2021**

Homework 5

Due: October 7 at 11:59 PM. Submit on Canvas.

Problem 1: Consider the following solution to our wave equation on a string:

$$y(x,t) = \mathrm{Re}\left[\tilde{y}(x,t)\right], \quad \tilde{y}(x,t) = A\mathrm{e}^{\mathrm{i}k(x-vt)} + A\mathrm{e}^{-\mathrm{i}k(x+vt)+\mathrm{i}\phi}. \tag{1}$$

Here A, k, and ϕ are positive and real constants, and v is the wave speed.

10 **1A:** Let's first check that $y(x,t)$ is in fact a solution to the wave equation.

- 1A.1. Plug in $\tilde{y}$ into the wave equation and confirm it is a solution.
- 1A.2. Why is y also a solution?[1]. You don't need to show an explicit calculation but justify your reasoning in a sentence or two.

15 **1B:** Assume $\phi = 0$.

- 1B.1. Show that $2\cos\theta = \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}$.
- 1B.2. Deduce that $y(x,t) = 2A\cos(kx)\cos(\omega t)$. What is the appropriate value of ω for which this equation is true?
- 1B.3. This form of $y(x,t)$ is called a standing wave, because the profile $y(x,t)$ is a product of a cosine in t times a spatial function. What are the wavelength and period of this standing wave?

10 **1C:** Now suppose that $kx_* = \pi/4$.

- 1C.1. Sketch $\tilde{y}(x_*, 0)$ (i.e. evaluate at $x = x_*, t = 0$) in the complex plane, assuming $\phi = 0$.
- 1C.2. Assuming $\phi = 0$, describe what happens to this complex number as a function of time. What is the consequence for the physical value of $y(x_*, t)$?
- 1C.3. For what value of ϕ is the amplitude of oscillations at $x = x_*$ as large as possible? (There is a nice and elegant graphical answer to this question!)

Problem 2 (Resonance): Consider a mass m, attached to a spring of spring constant k, pushed on by an external force $f(t)$. If the displacement of the spring is $x(t)$, we may write

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx + f(t). \tag{2}$$

Define the natural (angular) frequency of this harmonic oscillator to be

$$\omega = \sqrt{\frac{k}{m}}. \tag{3}$$

[1] *Hint:* Add y and its complex conjugate together. A complex conjugate function can be found by replacing $\mathrm{i} \to -\mathrm{i}$.

15 **2A:** Suppose that the external force $f(t)$ is itself oscillating with time:

$$f(t) = f_0 \cos(\varOmega t). \tag{4}$$

2A.1. Show that $f(t) = \mathrm{Re}(f_0 \mathrm{e}^{-\mathrm{i}\varOmega t})$. Note that f_0, being physical, is real-valued.

2A.2. Argue with a few words and/or equations that you can solve this differential equation by solving for $\tilde{x}(t)$, the solution to (2) with complex-valued driving force $f(t) = f_0 \mathrm{e}^{-\mathrm{i}\varOmega t}$.

2A.3. Let x_0 denote a constant (i.e. no t-dependence). Try to plug in the ansatz

$$\tilde{x}(t) = \tilde{x}_0 \mathrm{e}^{-\mathrm{i}\varOmega t} \tag{5}$$

into (2). Solve for $\tilde{x}_0$.

2A.4. Deduce the physical motion of the oscillator in response to the external force.

10 **2B:** Suppose that you drive the oscillator with a more complicated function of time, such as

$$f(t) = f_1 \cos(\varOmega_1 t) + f_2 \cos(\varOmega_2 t) + f_3 \cos(\varOmega_3 t). \tag{6}$$

2B.1. Show that one solution to (2) corresponds to the sum of three solutions $x_1 + x_2 + x_3$, with e.g. x_1 the response to driving f_1.

2B.2. Suppose that $f_1 = f_2 = f_3$, while $\varOmega_1 = 0.6\omega$, $\varOmega_2 = 0.99\omega$, and $\varOmega_3 = 1.6\omega$. To good approximation, $x(t)$ is driven at a single frequency. What is it, and why?

The phenomenon of oscillators preferentially responding to external forces at specific frequencies is known as **resonance**, and we will see this phenomenon arise again in our study of waves.

25 **Problem 3 (Torsion waves):** Consider a rod which is being twisted, as depicted in Figure 1.

The moment of inertia per unit length of the rod is J: namely, if we consider the segment of the rod stretching from x to $x + \mathrm{d}x$, it has a moment of inertia $J\mathrm{d}x$. Further assume that if two adjacent segments of a rod have been twisted by a *relative angle* of $\theta(x + \mathrm{d}x) - \theta(x)$, that there is a torque on the segment of length $\mathrm{d}x$ of magnitude (here $\kappa > 0$ is a constant, which is a material property of the rod)

$$\tau = \kappa \frac{\theta(x + \mathrm{d}x) - \theta(x)}{\mathrm{d}x} \tag{7}$$

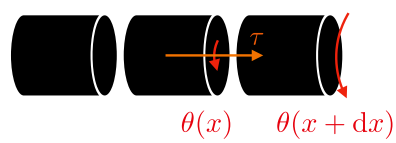

Figure 1: A rod being twisted; the local twisting creates internal torques.

3.1. Analogous to Lecture 15, by considering both the segment to the left and the right of the segment at point x, determine the total torque on a segment.

3.2. Using the angular analogue of Newton's Second Law, write down an equation for $\partial^2\theta/\partial t^2$.

3.3. Deduce that $\theta(x, t)$ obeys a wave equation. What is the speed of the waves in the rod?

15 **Problem 4 (Global Positioning System):** The Global Positioning System (or GPS) is an extremely accurate way of determining your location on Earth. It works by measuring the time it takes for light to propagate from your position on Earth to a network of satellites orbiting the Earth in orbits of radius R. It is sufficiently accurate that the effects of both special and general relativity must be taken into account when calculating how light propagates to and from the satellite.

In this problem, we will make some quick estimates for the relevance of both special and general relativity for the accuracy of GPS. Use the fact that Earth's mass is $M \approx 6 \times 10^{24}$ kg; Newton's gravitational constant is $G \approx 6.7 \times 10^{-11}$ $\mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$; the radius of the Earth is $r = 6.4 \times 10^6$ m; and the distance above Earth at which GPS satellites orbit is $R \approx 2 \times 10^7$ m.

4.1. If the GPS satellite is in a circular orbit above the Earth, how fast is it moving? Use non-relativistic mechanics to show that the satellite's speed is

$$v = \sqrt{\frac{GM}{R}}. \tag{8}$$

4.2. Deduce that the clock on the GPS satellite will therefore tick more slowly due to time dilation. If the clock ticking time was τ before time dilation, argue that the ticking time becomes $\tau(1+\delta_1)$, where $\delta_1 \ll 1$ is a small parameter you should calculate numerically.

4.3. The clock on the GPS satellite is also ticking at a different rate than it would on Earth due to the gravitational time dilation effect. Argue that if you only take this account into effect, then the ticking time becomes $\tau(1+\delta_2)$, where $|\delta_2| \ll 1$ is a small parameter that you should calculate.

4.4. Count the number of seconds in a day, and then estimate the time by which the satellite clock time would be "off" over the course of 1 day due to the errors accumulated above. Namely, how much time is $\Delta\tau = (|\delta_1| + |\delta_2|) \times 1$ day?

4.5. Since GPS works by sending signals from the satellite to your device, and then measuring the time delay it took for the signal to propagate, even tiny errors in the satellite clock become dangerous very quickly, and our error in position is

$$\Delta x = c\Delta\tau \tag{9}$$

if we only corrected for relativistic effects once a day. What is Δx, and is the effect substantial enough that you would expect current GPS satellites to actively try to account for relativistic effects?

15 **Problem 5 (Energy transport in a wave):** One of the important properties of waves, which we do not have time to cover in lectures, is the transport of energy, momentum, etc. For a wave on a string, it is most natural to think about the transport of energy, which should indeed be conserved.

In general, when we have a conserved quantity in a continuous medium (in this case energy E), we want to write it as

$$E = \int \mathrm{d}x \; \epsilon(x,t), \tag{10}$$

where $\epsilon(x,t)$ is the **energy density** of the system: $\epsilon(x,t)\mathrm{d}x$ is the energy stored in the string from x to $x+\mathrm{d}x$, at time t. Because physics is local in space (no teleportation), we must be able to write

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial J}{\partial x} = 0, \tag{11}$$

for an **energy current** $J(x,t)$.

5.1. Justify (11) by evaluating the change in the energy stored in the domain $x_1 \le x \le x_2$, and showing that energy can only enter/exit the region at the boundary. This means that energy is locally conserved in the interior of the region.

5.2. For our particular example of the wave on a string (with tension T and mass density μ, and displacement y), argue that[2]

$$\epsilon = \frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2. \tag{12}$$

5.3. Find an energy current J such that when the wave equation holds, (11) is satisfied.

[2] *Hint:* Use work-energy relations to estimate the potential energy stored when a strung under tension is bent.