PHYS 2170

General Physics 3 for Majors

Fall 2021

Lecture 11

Relativistic forces

September 17

1 Use energy conservation to argue for a relativistic definition of force.

$$\frac{d}{dt}\left[\begin{matrix}\text{energy}\\ \text{of 1 particle}\end{matrix}\right] = \frac{d}{dt}\left[\underbrace{\text{rest energy} + \text{kinetic energy}} + \underbrace{\text{potential energy}}_{U}\right] = 0$$

$E^2 = (pc)^2 + (mc^2)^2$

$$E = \gamma mc^2$$
$$= \sqrt{(mc^2)^2 + (pc)^2}$$

$$\frac{dE}{dt} = \frac{2c^2 p \, dp/dt}{2\sqrt{(mc^2)^2 + (pc)^2}}$$

$$= \boxed{\frac{c^2 p}{E}} \frac{dp}{dt}$$

Since $p = \gamma m u$

$E = \gamma mc^2$

$$\frac{c^2 p}{E} = \frac{\cancel{\gamma}\cancel{m}u\cancel{c^2}}{\cancel{\gamma}\cancel{m}\cancel{c^2}}$$
$$= u$$

$$\frac{dU(x)}{dt} = \underbrace{\frac{dU}{dx}}_{-F}\overbrace{\frac{dx}{dt}}^{u} \quad \text{(velocity)}$$

$$\frac{d}{dt}(E + U) = 0$$

$$= \underbrace{u\left(\frac{dp}{dt} - F\right)}$$

$= 0$

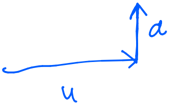
$$\boxed{F = \frac{dp}{dt}}$$

2 What does the analogue of $F = ma$ look like?

$$\vec{u} = \frac{d\vec{x}}{dt} \qquad \vec{a} = \frac{d\vec{u}}{dt}$$

If $\vec{p} = \gamma m \vec{u}$ $\qquad \vec{F} = \frac{d\vec{p}}{dt}$

(p_x, p_y, p_z) $\quad (u_x, u_y, u_z)$



(circular motion)

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\left[\gamma m \vec{u}\right] = m\left[\frac{d\gamma}{dt}\vec{u} + \gamma \frac{d\vec{u}}{dt}\right]$$

$$= m\left[0 + \underline{\gamma \vec{a}}\right]$$

for perpendicular $\vec{F} \perp \vec{u}$:

$$\vec{F} = \gamma m \vec{a}$$

$\vec{x}$ (linear motion)

$$\frac{d\vec{p}}{dt} = \gamma m \vec{a} + m\vec{u}\frac{d\gamma}{dt}$$

$$\frac{d}{dt}\left[1 - \frac{u^2}{c^2}\right]^{-1/2} = \frac{-\frac{1}{2}\left(-\frac{2u}{c^2}\frac{du}{dt}\right)}{\left(1 - u^2/c^2\right)^{3/2}}$$

$$= \frac{u}{c^2}\frac{du}{dt}\gamma^3$$

$$(\gamma m a) \cdot \frac{1 - u^2/c^2}{1 - u^2/c^2}$$

$$F = \gamma^3 m a\cancel{\left(1 - \frac{u^2}{c^2}\right)} + \cancel{\gamma^3 m u \frac{u}{c^2} a}$$

$$= \gamma^3 m a$$

3 Describe the motion of charged particles in magnetic fields.

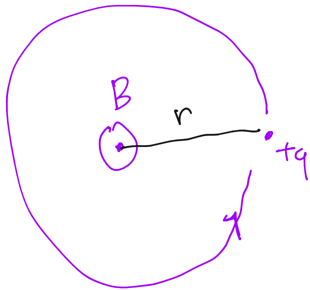
$\vec{F} = q\vec{u} \times \vec{B}$: true in relativity

$a_c = \frac{u^2}{r}$: also true

$\vec{a} \perp \vec{u}$

$\boxed{\vec{F} \perp \vec{u}}$ $\qquad \vec{F} \perp \vec{p}$

$\vec{u} = \frac{d\vec{x}}{dt} \qquad \vec{a} = \frac{d\vec{u}}{dt}$



Thus

$$\vec{F} = \frac{d\vec{p}}{dt} = \gamma m \vec{a} = q\vec{u} \times \vec{B}$$

$$\gamma m a_c = \gamma m \frac{u^2}{r} = quB$$

(magnitude of vectors)

$$\gamma m u = qBr$$

$$r = \frac{p}{qB}$$

4 When is it reasonable to use $F = ma$?

$$F_\perp = \gamma m a_\perp \qquad F_\parallel = \gamma^3 m a_\parallel$$

$\vec{F} \simeq m\vec{a}$ when $|\vec{u}| \ll c$

Estimate when $F = ma$ off by 1%?

$\gamma \simeq 1.01$

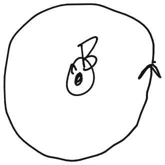
Taylor expand:

$$\gamma = \left[1 - v^2/c^2\right]^{-1/2} \simeq 1 + \underbrace{\frac{1}{2}\frac{v^2}{c^2}}_{1/100}$$

(usually) reasonable to think in Newtonian below this velocity.

$$\frac{v}{c} = \sqrt{\frac{2}{100}} = \frac{\sqrt{2}}{10} \qquad v \simeq 4 \times 10^7 \text{ m/s}$$

5 A naive physicist applying Newtonian mechanics observes a charged particle moving in an orbit, in a magnetic field, of radius such that it seems the particle travels at speed c. What is the actual speed?



If Newtonian $[F=ma]$

$$v_{Newt} \approx c$$

How fast is particle moving?

$$a_c = \frac{u^2}{r}$$

$$m a_c = quB$$

$$\downarrow$$

$$r = \frac{mu}{qB}$$

$$r = \frac{mc}{qB}$$

$$r = \frac{\gamma m u}{qB} = \frac{mc}{qB}$$

$$\gamma \frac{u}{c} = 1 \qquad \frac{u}{c} = \beta$$

$$\frac{\beta}{\sqrt{1-\beta^2}} = 1 \qquad \beta^2 = 1 - \beta^2$$

$$2\beta^2 = 1 \qquad \boxed{\beta = \frac{1}{\sqrt{2}}}$$