PHYS 2170

General Physics 3 for Majors

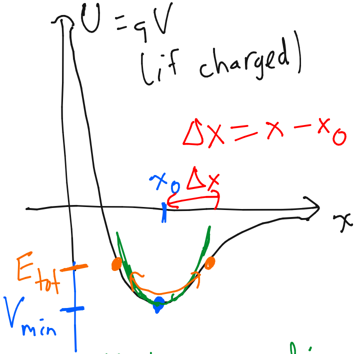
Fall 2021

Lecture 13

Harmonic oscillators and complex numbers

September 22

1 Motivate the ubiquity of harmonic oscillation.



$$F = ma = \boxed{-kx}$$

Taylor expand: $x = x_0$

$$V(x) = \underbrace{V(x_0)}_{V_{min}} + \cancel{\underbrace{V'(x_0)}_{=0}(x-x_0)} + \overbrace{\frac{1}{2}V''(x_0)}^{k\text{: "effective" spring constant,}}\overbrace{(x-x_0)}^{\Delta x}{}^2$$

$$F = ma = m\frac{d^2\Delta x}{dt^2} = -\frac{dV(x)}{dx} = -\frac{d}{dx}\left[V(x_0) + \frac{1}{2}V''(x_0)(x-x_0)^2\right]$$

$$= \cancel{0} - V''(x_0)\overbrace{(x-x_0)}^{\Delta x}$$

2 Describe the solution of linear differential equations with constant coefficients. $(x_0 = 0)$

$$m\underbrace{\frac{d^2x(t)}{dt^2}} = -\underbrace{k}\,x(t)$$

const. Example:

$$\frac{dx(t)}{dt} = cx(t)$$

const.

Answer: $x(t) = Ae^{ct}$

Check: $x(t) = e^{bt}$ "Ansatz"

$$\frac{dx}{dt} = be^{bt} = bx$$

"$\frac{d}{dt} \to b$"

$$\frac{dx(t)}{dt} = bx = cx$$

True: if $b = c$.

e^{ct} is sol'n: multiply by const.

Example:

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0.$$

Plug in $e^{bt} = x(t)$:

$$b^2e^{bt} - 3be^{bt} + 2e^{bt} = 0$$

Hold for all t:

$$\cancel{e^{bt}}(b^2 - 3b + 2) = 0$$

$$(b-1)(b-2) = 0$$

$\boxed{b = 1 \text{ or } b = 2}$

Most general:

$$x(t) = A_1 e^{t} + A_2 e^{2t}$$

$b=1$, $b=2$

const.

$$\frac{dx}{dt} = A_1 \cdot 1e^{t} + A_2 \cdot 2e^{2t}$$

3 Try to solve the harmonic oscillator in this way.

$$m\frac{d^2x}{dt^2} = -kx$$

$$\omega = \sqrt{\frac{k}{m}}$$

$k>0$

$m>0$

$\omega>0$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Try:

$$x(t) = e^{bt}$$

$$\frac{d^2x}{dt^2} = b^2 e^{bt} = -\omega^2 e^{bt}$$

$$b^2 = -\omega^2$$

Define $\boxed{i=\sqrt{-1}}$

$$b^2 = (-1)\omega^2$$

$$= i^2\omega^2$$

$$= (i\omega)^2$$

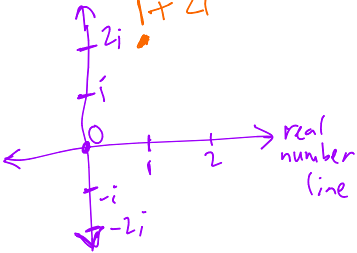
$$b = \pm i\omega$$

$(i\omega$ or $-i\omega)$

Re Im

(1, 2)

$1+2i$



$$(1+2i)^2$$

$$= 1-4+4i$$

$$= -3+4i$$

4 ~~Explain~~ $\underbrace{e^{i\theta} = \cos\theta + i\sin\theta.}_{\text{Take as given.}}$

$x(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$

$x(t)$ is physical

- anything measurable must be real

$x + 0i$

$= A_1[\cos(\omega t) + i\sin(\omega t)] + A_2 e^{-i\omega t}$

$+ A_2[\cos(\omega t) - i\sin(\omega t)]$

$= \underbrace{(A_1 + A_2)}_{=C_1}\cos(\omega t) + \underbrace{i(A_1 - A_2)}_{=C_2}\sin(\omega t)$

$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$

must be real: $A_1 = \overset{Re}{\alpha} + i\overset{Im}{\beta}$, $A_2 = \alpha - i\beta$ } complex conjugates

5 Describe the harmonic oscillator in terms of a phasor.