PHYS 2170

General Physics 3 for Majors

Fall 2021

Lecture 14

Euler's formula

September 24

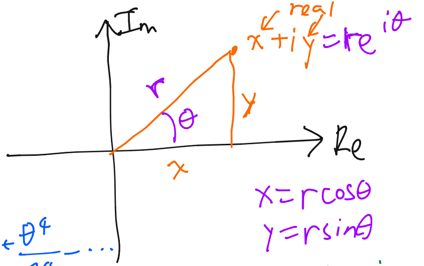
1 Explain $e^{i\theta} = \cos\theta + i\sin\theta$.

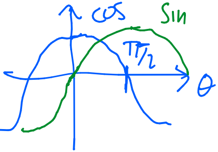
Method #1: Taylor series

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\theta)^n$$

$$\cos\theta = \sum_{n=0,2,4,\dots} \frac{1}{n!}(-1)^{n/2}\theta^n = \underbrace{1-\frac{\theta^2}{2}}+\frac{\theta^4}{24}-\dots$$

$$\sin\theta = \sum_{n=1,3,5,\dots} \frac{1}{n!}(-1)^{\frac{n-1}{2}}\theta^n = \underbrace{\theta}-\frac{\theta^3}{6}+\dots$$





$(-1) = i^2$

Since $i^2 = -1$; $e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{3!} + \dots$

$$= \underbrace{\cos\theta}_{\text{real}} + i\underbrace{\sin\theta}_{\text{imaginary}}$$

Method #2:

$\frac{d}{d\theta}e^{i\theta} = i e^{i\theta}$; $\frac{d}{d\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta$

$= i^2\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta)$

$\boxed{2}$ All physical quantities must be real when using complex numbers to solve a physical problem.

$F = ma = -kx$

$m\frac{d^2x}{dt^2} = -kx$

let $\omega = \sqrt{\frac{k}{m}}$,

$\frac{d^2x}{dt^2} = -\omega^2 x$

Solve a linear eqn $x = e^{bt}$ (b unknown)

plug in

$b^2 \cancel{e^{bt}} = -\omega^2 \cancel{e^{bt}}$

$b = +i\omega$ or $-i\omega$

$x(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$

A_1, A_2 unknown constants.

$\underline{\text{PHYSICAL}}$ = $\underline{\text{REAL}}$

(measurable)

$x(t) = (A_1 + A_2)\overbrace{\cos(\omega t)}^{\text{real}}$
$+ i(A_1 - A_2)\overbrace{\sin(\omega t)}^{\text{real}}$

Thus $A_1 + A_2$ real

$i(A_1 - A_2)$ real

$A_1 = \alpha + i\beta$ (real)

$A_2 = \alpha - i\beta$

complex conjugate

$A_1 + A_2 = \alpha + \cancel{i\beta}$
$+ (\alpha - \cancel{i\beta})$

$\boxed{3}$ Describe the motion of a harmonic oscillator in terms of a "phasor" in the complex plane.

$x(0) = A$

Solution: $x(t) = \overset{A}{\cancel{a_1}} \cos(\omega t) + \cancel{a_2 \sin(\omega t)}$

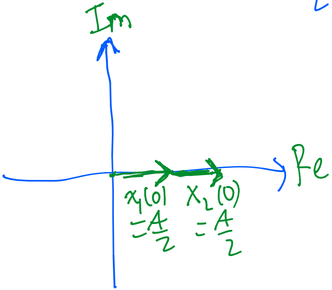
$$\left.\frac{dx}{dt}\right|_{t=0} = \omega\left[\cancel{-a_1 \sin} + a_2 \cancel{\cos}\right] = 0$$

$a_2 = 0$

Complex: $x(t) = \underbrace{\frac{A}{2} e^{i\omega t}}_{x_1} + \underbrace{\frac{A}{2} e^{-i\omega t}}_{x_2}$.

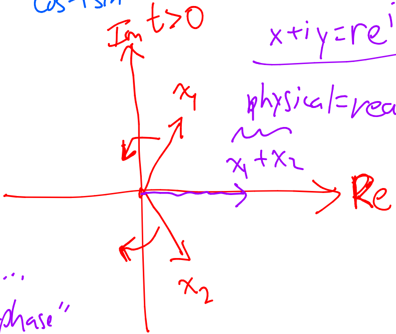
$e^{i\omega t}$: $\cos + i\sin$ $\qquad$ $e^{-i\omega t}$: $\cos - i\sin$

$e^{i\omega \cdot 0} = 1 = e^{-i\omega \cdot 0}$



$x + iy = re^{i\theta}$

physical = real



$x_1(t)$ "rotates" in time because...

$x_1 = \frac{A}{2} e^{i\omega t}$ ... complex "phase" is ωt, amplitude $A/2$ is fixed.

4 Define amplitude, period, frequency and angular frequency.

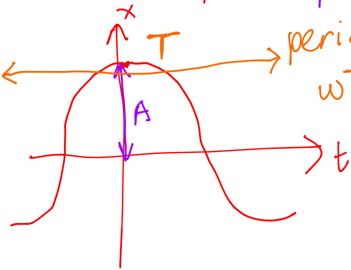
Claim: $\underbrace{x(t)}_{\text{physical sol'n to oscillator problem}} = \mathrm{Re}\left[\tilde{x}(t)\right]$ ← aux. complex.

$x(t) = A\cos(\omega t)$

$\tilde{x}(t) = -Ai\sin(\omega t) + A\cos(\omega t)$

$= Ae^{-i\omega t}$

"Old" way:



amplitude: A

period: T

$\omega T = 2\pi$

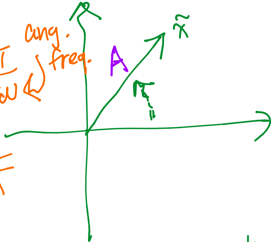
$T = \frac{2\pi}{\omega}$ ← ang. freq.

$f = \frac{1}{T}$ ← frequency

$x(t) = A\cos(\omega t)$

$x(t) = x(t+T)$

"New way":



$\tilde{x}(t) = Ae^{-i\omega t}$

END of calc: $x(t) = \mathrm{Re}[\tilde{x}]$

5 Describe the response of a harmonic oscillator driven by a sinusoidal forcing $f(t) = f_0 \cos(\Omega t)$.

$$m\frac{d^2x}{dt^2} = -kx + f_0\cos(\Omega t) \qquad \omega = \sqrt{\frac{k}{m}}$$

$$x = \mathrm{Re}(\tilde{x}) \qquad a_0 = \frac{f_0}{m}$$

$$\cos(\Omega t) = \mathrm{Re}\left(e^{-i\Omega t}\right)$$

$$\frac{d^2\tilde{x}}{dt^2} = -\omega^2\tilde{x} + a_0 e^{-i\Omega t}$$

$$\tilde{x} = z e^{-i\Omega t}$$

Plug in:

$$e^{-i\Omega t} z \times (-\Omega^2) = -\omega^2 z e^{-i\Omega t} + a_0 e^{-i\Omega t}$$

$$(-\Omega^2) = (-i\Omega)^2$$

divide by $e^{-i\Omega t}$

$$-z\Omega^2 = -\omega^2 z + a_0$$

$$z = \frac{a_0}{-\Omega^2 + \omega^2}$$

$$x = \mathrm{Re}(\tilde{x}) = \mathrm{Re}\left(e^{-i\Omega t} z\right) = \frac{a_0}{\omega^2 - \Omega^2}\cos(\Omega t)$$