PHYS 2170

General Physics 3 for Majors

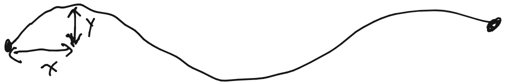
Fall 2021

Lecture 16

The wave equation

October 1

1 Review the wave equation on a string.



- mass per unit length μ
- tension T

Equation for $y(x,t)$:

$$T\frac{\partial^2 y}{\partial x^2} = \mu\frac{\partial^2 y}{\partial t^2}$$

$$\frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

$v=\sqrt{\frac{T}{\mu}}$ is wave speed

$$\boxed{v^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}}$$

"<u>the</u> wave equation"

2 Try to solve the wave equation using an exponential ansatz. What do you find?

$$v^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

- LINEAR: y shows up "once" in each term
- v is a constant (no x or t dependence)

add two sol'ns → new solution (superposition)

$$y(x,t) = e^{-i\omega t + ikx}$$

(k & ω "unknown")

$$\frac{\partial y}{\partial x} = e^{-i\omega t + ikx} \cdot ik = iky$$

$$\frac{\partial y}{\partial t} = -i\omega y$$

$$\frac{\partial^2 y}{\partial x^2} = ik\frac{\partial y}{\partial x} = (ik)^2 y = -k^2 y$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 y$$

$$v^2(-k^2)y = -\omega^2 y$$

$$\omega^2 = v^2 k^2 \quad \text{or} \quad \omega = \pm vk$$

No constraint on k:

$$y(x,t) = c_1 e^{ik_1(x-vt)} + c_2 e^{ik_2(x+vt)} + \cdots$$

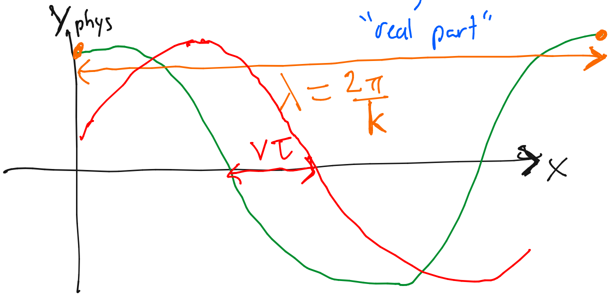
$$y(x,t) = \int dk \, A(k) e^{ik(x-vt)} + B(k) e^{ik(x+vt)}$$

3 Describe a traveling wave. What is its wavelength and period?

"Building blocks" are $e^{i(kx - \omega(k)t)}$ $\qquad e^{i\theta} = \cos\theta + i\sin\theta$

physical response is real: $\mathrm{Re}\left[e^{ik(x-vt)}\right] = \cos(k(x-vt))$

"real part"



$t=0$: $y_{phys}(x,0) = \cos(kx)$

$\cos(k \cdot \frac{2\pi}{k}) = \cos(2\pi) = 1$

at time $t = \tau$

$\cos(k(x - (v\tau)))$

λ = wavelength

v = speed of wave

τ = period of oscillations $= \frac{\lambda}{v}$: cosine shift by $v \cdot \frac{\lambda}{v} = \lambda$

4 Show that the wave equation's general solution takes the form $y(x,t) = y_1(x - vt) + y_2(x + vt)$.

Our general sol'n: $y = \mathrm{Re}\left[\int dk \left\{ A(k) e^{\underbrace{ik(x-vt)}} + B(k) e^{\underbrace{ik(x+vt)}} \right\}\right]$

$= \tilde{A}(x-vt)$ right-moving

$\tilde{B}(x+vt)$ left-moving

Claim: all sol'ns take form

$y = y_1(x-vt) + y_2(x+vt)$

arbitrary func

$$\frac{\partial y}{\partial t} + v\frac{\partial y}{\partial x} = -vy_1' + vy_1' = 0$$

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0$$

$$\frac{\partial y}{\partial t} - v\frac{\partial y}{\partial x} = 0 \Rightarrow y = y_2(x+vt)$$

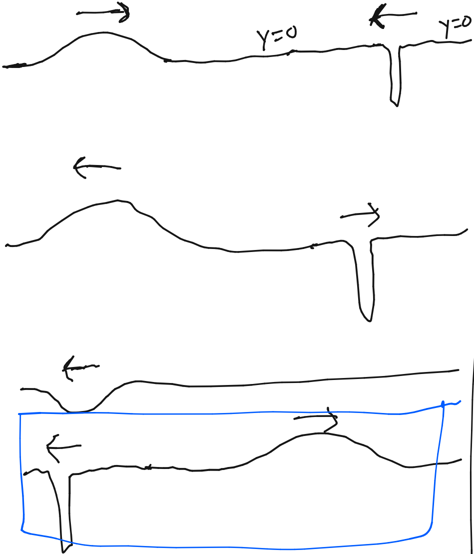
$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) y = 0$$

$(\partial/\partial t + v\,\partial/\partial x)\,y = 0\,?$ $\Rightarrow$ $y = y_1(x-vt)$

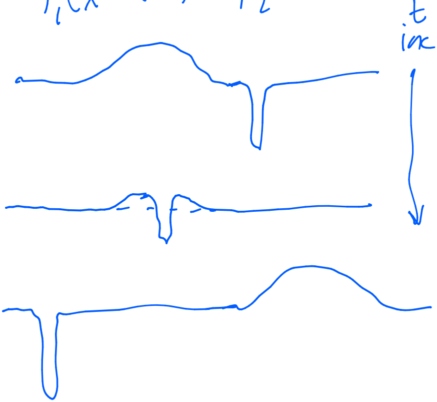
$$\frac{\partial y}{\partial t} = -v\,y_1'(x-vt), \quad \frac{\partial y}{\partial x} = y_1'(x-vt)$$

5 Use the linearity of the wave equation to describe how waves travel through each other as they propagate.

What happens if:



$y_1(x-vt) + y_2(x+vt)$



"principle of superposition"