PHYS 2170

General Physics 3 for Majors

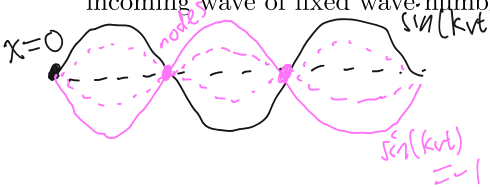
Fall 2021

Lecture 17

Standing waves

October 6

1 Suppose that we tie the end of a string (that exists for $x \geq 0$) down at $x = 0$. What is the boundary condition on y? What happens to an incoming wave of fixed wave number k?



incoming / left-moving wave

$$g(x+vt) = e^{-ik(x+vt)}$$

Last time: reflected wave $f(x-vt)$

$$f(x) = -g(-x)$$

$(\theta = kx)$

$$e^{-i\theta} - e^{i\theta}$$
$$= (\cancel{\cos\theta} - i\sin\theta) - (\cancel{\cos\theta} + i\sin\theta)$$
$$= -2i\sin\theta$$

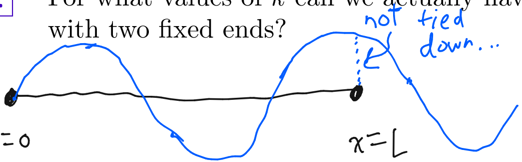
$$y(x,t) = f(x-vt) + g(x+vt)$$
$$= -e^{-ik(-(x-vt))} + e^{-ik(x+vt)}$$
$$= e^{-ikx-ikvt} - e^{ikx-ikvt}$$
$$= \underbrace{e^{-ikvt}}_{(\cos - i\sin(kvt))}\underbrace{(e^{-ikx} - e^{ikx})}_{= -2i\sin(kx)}$$

Physical response:

$$y(x,t)_{phys} = -2\sin(kx)\sin(kvt)$$

standing wave

2 For what values of k can we actually have a standing wave on a string with two fixed ends?



If endpoint at $x=L$ not there

$$y = \underbrace{\sin(kx)}\, e^{-ivkt}$$

$$e^{-i\omega t} = e^{-2\pi i f t}$$

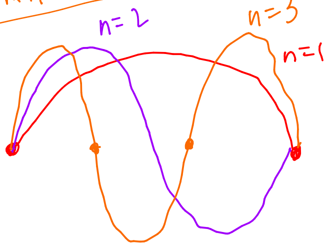
If we want $y(x=L, t) = 0 \ldots$

$$\sin(kL) = 0$$

$$kL = n\pi \qquad n = 1, 2, 3, \ldots$$

$$kL = \cancel{0}, \boxed{\pi, 2\pi, 3\pi, \ldots}, -\pi, -2\pi, \ldots$$

normal modes



[if $k=0$

$$\sin(k\cdot x) = \sin(0\cdot x) = 0$$

$$\sin(kx) = -\sin((-k)x)$$

n^{th} normal mode:

- wavelength: $\lambda_n = 2L/n$
- angular frequency: $\omega_n = vk_n = \frac{n\pi}{L}v$

($1\ Hz = 1\ s^{-1}$) frequency: $f_n = \frac{n}{2}v/L = \frac{1}{2\pi}\omega_n$

3 Find the standing waves (normal modes) of a string with two fixed endpoints by directly solving the wave equation.

Another strategy:

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

if $y(x,t)$ sol'n, $c \cdot y(x,t)$ also sol'n.

guess: $y(x,t) = \underbrace{e^{-i\omega t} Y(x)}$

Wave eqn:

$$\frac{\partial^2 y}{\partial t^2} = \overset{(-i\omega)^2}{-\omega^2} Y e^{-i\omega t}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2 Y}{dx^2} e^{-i\omega t}$$

$$\Rightarrow \quad v^2 \frac{d^2 Y(x)}{dx^2} = -\omega^2 Y(x)$$

Try $Y(x) = e^{ikx}$? $-v^2 k^2 = -\omega^2$ $k = +\frac{\omega}{v}$ or $-\frac{\omega}{v}$

General: $Y(x) = A_1 e^{i\omega x/v} + A_2 e^{-i\omega x/v}$

$A_1 = 0$ [bad]

$$\boxed{e^{2i\omega L/v} = 1}$$

$Y(0) = Y(L) = 0$

$Y(0) = 0 = A_1 \cdot 1 + A_2 \cdot 1 \Rightarrow A_1 = -A_2$

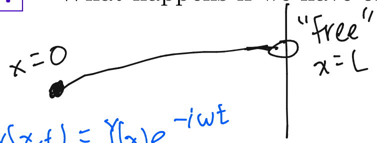
$Y(L) = 0 = A_1 e^{i\omega L/v} - A_1 e^{-i\omega L/v} = 0$

$n = 0, \pm 1, \ldots$

$$\frac{2\omega L}{v} = 2\pi n$$

$$\boxed{\omega_n = \frac{n\pi v}{L}}$$

$\boxed{4}$ What happens if we have one fixed end and one free end?



$$\begin{cases} y(x=0,t)=0 \\ \quad\text{and} \\ \frac{\partial y}{\partial x}(x=L,t)=0 \end{cases}$$

$$y(x,t) = Y(x)e^{-i\omega t}$$

$$Y(x) = A_1 e^{i\omega x/v} + A_2 e^{-i\omega x/v}$$

$$Y(0) = 0 \Rightarrow A_1 = -A_2$$

$$Y'(L) = 0 = \frac{i\omega}{v}A_1 e^{i\omega L/v} + (-A_1)\left(-\frac{i\omega}{v}\right)e^{-i\omega L/v}$$

$$0 = e^{i\omega L/v} + e^{-i\omega L/v} = 2\cos\frac{\omega L}{v}$$

$$\frac{\omega L}{v} = \frac{\ell\pi}{2} \qquad \ell = 1, 3, \ldots 5, \ldots$$

- Wavelengths of standing wave $\lambda = 4\frac{L}{\ell}$

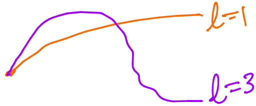
$$2\pi f_\ell = 2\pi\frac{v}{\lambda_\ell}$$

$$\uparrow$$

$$\omega_\ell = v k_\ell$$

$$\frac{\ell\pi v}{2} = v\frac{2\pi}{\lambda_\ell}$$

$$\lambda_\ell f_\ell = v$$



$\boxed{5}$ What happens if we have two free ends?