PHYS 2170

General Physics 3 for Majors

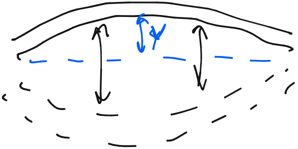
Fall 2021

Lecture 21

Phase and group velocity

October 13

1 Review flexural waves in metal rods.



flexural wave

$$\frac{\partial^2 y}{\partial t^2} = -a^2 \frac{\partial^4 y}{\partial x^4}$$

$$a \sim \sqrt{\frac{Y}{\rho}} \cdot w$$

$\sqrt{Y/\rho}$: speed of sound

w: width of rod

Plug in guess:

$$y(x,t) = e^{ikx - i\omega t}$$

$$\frac{\partial y}{\partial t} = -i\omega \cdot y \quad \text{(e.g.)}$$

$$(-i\omega)^2 y = -a^2 (ik)^4 y$$

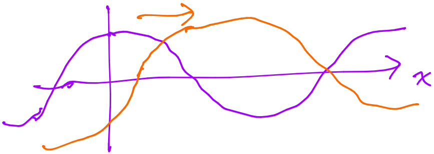
$$-\omega^2 = -a^2 k^4$$

dispersive waves

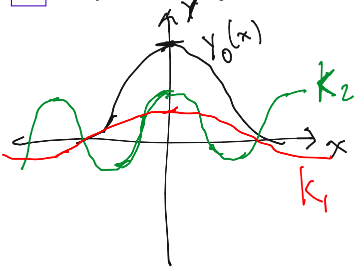
$$\boxed{\omega = \pm a k^2}$$

"plane wave" $\left(e^{ikx - i\omega t}\right)$

physical ↓ $\mathrm{Re}[y]$



2 Qualitatively describe the difference between phase and group velocity.



$$Y_0(x) = \sum e^{ik_1 x} c_1 + e^{ik_2 x} c_2 + \dots$$

$$Y_0(x) = \int dk\, c(k)\, e^{ikx}$$

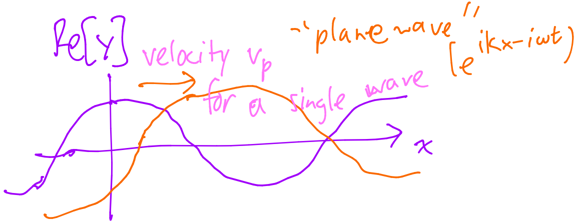
time evol. →

$$y(x,t) = \int dk\, c(k)\, e^{ikx - i\omega(k)t}$$

- local "wave packet" = sum of many sine/cosine waves
- each wave moves @ vel. $v_p(k)$
- how does $y(x,t)$ evolve?

$$e^{ikx - i\omega t} = e^{ik(x - v_p t)}$$

$$v_p = \frac{\omega(k)}{k}$$



3 Derive a formula for the group velocity of waves. $\omega_2 = \omega(k_2)$

Toy problem: $y(x,t) = e^{i(k_1 x - \omega_1 t)} + e^{i(k_2 - \omega_2 t)}$

Assume $k_1 \approx k_2$: $(k_+ \gg k_-)$

$k_+ = \frac{k_1 + k_2}{2}$ $\quad \omega_+ = \frac{\omega_1 + \omega_2}{2}$

$k_- = \frac{k_1 - k_2}{2}$ $\quad \omega_- = \frac{\omega_1 - \omega_2}{2}$

outer "envelope" cosine: $\omega(k_2) - \omega(k_1)$

$v_g = \frac{\omega_-}{k_-} = \frac{\omega_2 - \omega_1}{k_2 - k_1}$

↑ group velocity

$k_2 = k_1 + dk$

$$v_g = \frac{d\omega}{dk}\Big|_{k_+}$$

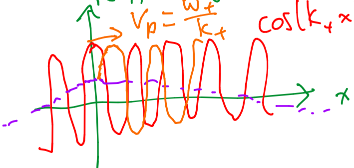
$$y(x,t) = \underbrace{e^{i(k_+ + k_-)x - i(\omega_+ + \omega_-)t}}_{\text{wave!}} + e^{i(k_+ - k_-)x - i(\omega_+ - \omega_-)t}$$

$$= e^{ik_+ x - i\omega_+ t}\left(e^{ik_- x - i\omega_- t} + e^{-ik_- x + i\omega_- t}\right)$$

$$= e^{ik_+ x - i\omega_+ t}\left(2\cos(k_- x - \omega_- t)\right)$$

$$\mathrm{Re}(y) = 2\cos(k_- x - \omega_- t) \cdot \cos(k_+ x - \omega_+ t)$$

$\cos(k_+ x)$ "avg plane wave"



multiply together:

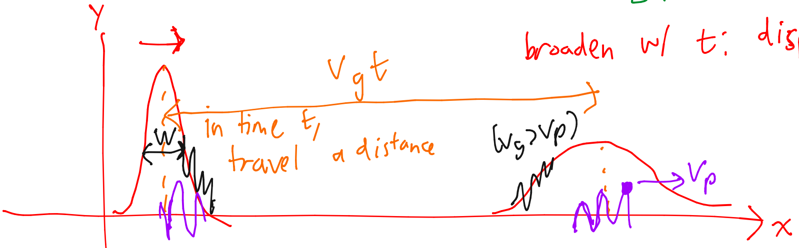
← $\cos(k_- x)$

4 For flexural waves, is phase or group velocity larger? At which velocity are signals sent?

Flex waves: $\omega(k) = ak^2$

$v_p = \frac{\omega(k)}{k} = \frac{ak^2}{k} = ak$ $\quad v_g = \frac{d\omega}{dk} = a \cdot 2k = 2ak$ $\quad \frac{v_g}{v_p} = 2$

broaden w/ t: dispersive



typical wave #: $\frac{2\pi}{w} \sim k_{typ}$

5 Which of the nice features of waves are true for general dispersive waves, vs. only holding when the "wave equation" is satisfied?