PHYS 2170

General Physics 3 for Majors

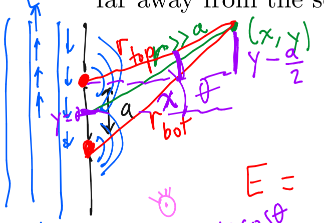
Fall 2021

Lecture 25

Diffraction

October 22

1 Light is incident on a slit with two holes in it. What is the image seen far away from the screen?



Intensity: $I = \frac{\varepsilon_0}{2}|E|^2$

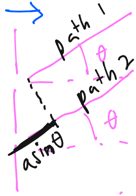
if $E = E_r + iE_i$

$|E|^2 = E_r^2 + E_i^2$

$|e^{ia}| = \cos^2 a + \sin^2 a = 1$

$\left(k = \frac{2\pi}{\lambda}\right)$

$$E = \frac{1}{\sqrt{r_{top}}} e^{ik(r_{top} - ct)} + \frac{1}{\sqrt{r_{bot}}} e^{ik(r_{bot} - ct)}$$



$x = r\cos\theta$

$y = r\sin\theta$

$$r_{top} = \sqrt{x^2 + \left(y - \frac{a}{2}\right)^2} \simeq \sqrt{x^2 + y^2 - ay + \cancel{\left(\frac{a}{2}\right)^2}}$$

$$= \sqrt{r^2 - ar\sin\theta}$$

$$= r\sqrt{1 - \frac{a}{r}\sin\theta} \simeq r\left(1 + \frac{1}{2}\left(-\frac{a}{r}\sin\theta\right)\right)$$ (comes of Taylor expansion)

$$\simeq r - \frac{1}{2}a\sin\theta$$

$$r_{bot} \simeq r + \frac{1}{2}a\sin\theta$$

$$E \simeq \frac{1}{\sqrt{r}}\left[e^{ik(r - \frac{1}{2}a\sin\theta - ct)} + e^{ik(r + \frac{a\sin\theta}{2} - ct)}\right]$$

$$\simeq \frac{1}{\sqrt{r}} \underbrace{e^{ik(r-ct)}}_{\cos(\,) + i\sin(\,)} \underbrace{\left[e^{-i\frac{ka}{2}\sin\theta} + e^{i\frac{ka}{2}\sin\theta}\right]}_{= 2\cos\left(\frac{ka}{2}\sin\theta\right)}$$

scales as

$$I \sim |E|^2 \sim \cos^2\left(\frac{ka}{2}\sin\theta\right)$$

local maximum in θ:

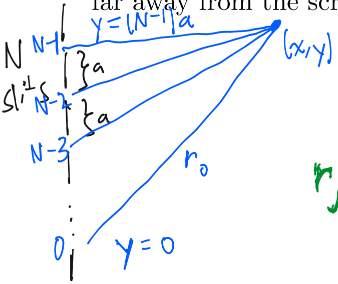
$\frac{ka}{2}\sin\theta = \pi n$ $(n = 0, 1, 2, \ldots)$

$\frac{\cancel{2\pi} a}{\cancel{2}\lambda}\sin\theta = \cancel{\pi} n$

$$\boxed{a\sin\theta = n\lambda}$$

2

Light is incident on a slit with N holes in it. What is the image seen far away from the screen?



$$E(x,y) = \frac{1}{\underbrace{\sqrt{r_0}}_{\simeq r}} e^{ik(r_0 - ct)} + \frac{1}{\underbrace{\sqrt{r_1}}_{\simeq r}} e^{ik(r_1 - ct)} + \cdots$$

$(j = 0, 1, \ldots, N-1)$

$$r_j = \sqrt{x^2 + (y - ja)^2}$$

$$\approx \sqrt{r^2 - 2r\sin\theta\, ja + \cancel{(ja)^2}} \simeq r - ja\sin\theta$$

$$E(x,y) = \sum_{j=0}^{N-1} \frac{1}{\sqrt{r}} e^{ik(r_j - ct)}$$

$$= \sum_{j=0}^{N-1} \frac{1}{\sqrt{r}} e^{ik(r-ct)} e^{-(ika\sin\theta)j}$$

$$E = \frac{1}{\sqrt{r}} e^{ik(r-ct)} \frac{e^{-ika\sin\theta \cdot N} - 1}{e^{-ika\sin\theta} - 1}$$

Math identity:

$$\sum_{j=0}^{N-1} z^j = \frac{z^N - 1}{z - 1}$$

3 Discuss the limit of a diffraction grating.

$ka\sin\theta = \alpha$

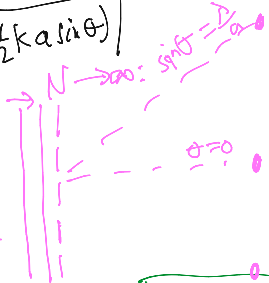
$$E = \frac{1}{\sqrt{r}} e^{ik(r-ct)} \underbrace{\frac{e^{-ika\sin\theta\cdot N}-1}{e^{-ika\sin\theta}-1}} = \frac{e^{-\frac{i\alpha N}{2}}}{e^{-i\alpha/2}} \frac{e^{-\frac{i\alpha N}{2}}-e^{+\frac{i\alpha N}{2}}}{e^{-i\alpha/2}-e^{i\alpha/2}}$$

$$I \sim |E|^2 \sim \left|\frac{e^{-i\alpha N/2}-e^{+i\alpha N/2}}{e^{-i\alpha/2}-e^{i\alpha/2}}\right| \qquad \left[e^{i\theta}-e^{-i\theta} = 2i\sin\theta\right]$$

$$\sim \left|\frac{\cancel{2i}}{\cancel{2i}} \frac{\sin(-\alpha N/2)}{\sin(-\alpha/2)}\right|^2 = \left|\frac{\sin\left(\frac{1}{2}ka\sin\theta\cdot N\right)}{\sin\left(\frac{1}{2}ka\sin\theta\right)}\right|^2$$

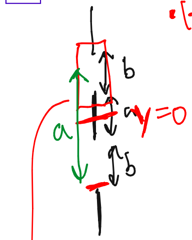
Sketch:

diffraction (grating)



θ_{bright}: $\frac{1}{2}ka\sin\theta_{bright} = \pi$ $\boxed{\lambda = a\sin\theta}$

4 What happens if the screens have finite width?



•(x,y)

$$E(x,y) = \sum_{\text{top}} \Delta y \frac{1}{\sqrt{r_y}} e^{ik(r_y-ct)} + \sum_{\text{bottom}} \Delta y \frac{1}{\sqrt{r_y}} e^{ik(r_y-ct)}$$

$$= \left(\int_{-\frac{b}{2}-\frac{a}{2}}^{\frac{b}{2}-\frac{a}{2}} + \int_{\frac{a}{2}-\frac{b}{2}}^{\frac{b}{2}+\frac{a}{2}} \right) dy \frac{1}{\sqrt{r_y}} e^{ik(r_y-ct)}$$

$$\int_c^d dz\, e^{\alpha z} = \frac{e^{\alpha d} - e^{\alpha c}}{\alpha}$$

$$r_y \approx r - y\sin\theta$$

$$E = \frac{e^{ik(r-ct)}}{\sqrt{r}} \underbrace{\left(\int + \int \right) e^{-ik\sin\theta\cdot y}}$$

$$= \frac{1}{-ik\sin\theta}\left[e^{-ik\left(\frac{a+b}{2}\right)\sin\theta} - e^{-ik\frac{a-b}{2}\sin\theta} + e^{ik\frac{a-b}{2}\sin\theta} - e^{ik\frac{a+b}{2}\sin\theta} \right]$$

as $b \to 0$:

1

$$I \sim \left| \cos\left(\frac{ka}{2}\sin\theta\right) \boxed{\frac{\sin\left(\frac{kb}{2}\sin\theta\right)}{k\sin\theta}} \right|^2$$