PHYS 2170

General Physics 3 for Majors

Fall 2021

Lecture 30

Uncertainty in position and momentum

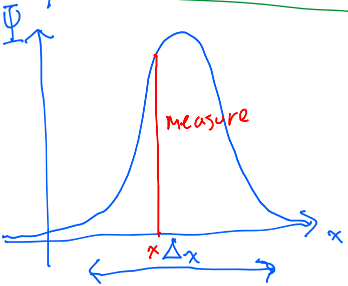
November 5

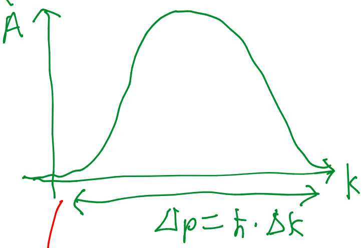
$\boxed{1}$ Review Heisenberg's uncertainty principle.

$$\boxed{\Delta x \, \Delta p \gtrsim \frac{\hbar}{2}}$$

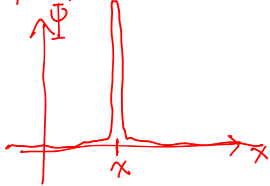
$$\Psi(x) = \int_{-\infty}^{\infty} dk \; e^{ikx} A(k)$$

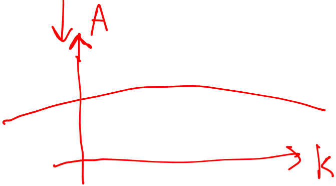
$$k = \frac{2\pi}{\lambda} = \frac{p}{\hbar}$$





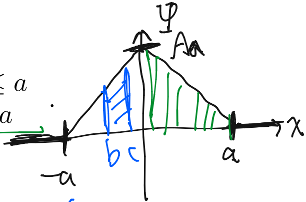
(Try to) measure x:





2 Consider the wave function

$$\Psi(x) = \begin{cases} A(a - |x|) & |x| \le a \\ 0 & x > a \end{cases}.$$



What is the normalization factor A?

Normalize : $\int_{-\infty}^{\infty} dx |\Psi|^2 = 1$

$\int_b^c dx\, |\Psi|^2$ prob.

$$\left[\int_{-\infty}^{-a} + \int_{-a}^{0} + \int_0^a + \int_a^\infty\right] dx\, |\Psi|^2$$

int: 0   0

$$A = \sqrt{\frac{3}{2a^3}}$$

$$= 2\int_0^a dx\, |A|^2 (a-x)^2 = 2\int_a^0 (-du)\, |A|^2 u^2 = 2|A|^2 \frac{u^3}{3}\Big|_0^a$$

u^2

$u = a - x$
$du = -dx$

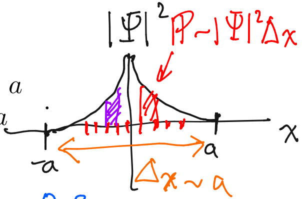
$$= \frac{2}{3} a^3 |A|^2 = 1$$

3 Consider the wave function

$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

for this: $\Delta x = \frac{a}{\sqrt{10}}$

$$\Psi(x) = \begin{cases} A(a-|x|) & |x| \le a \\ 0 & x > a \end{cases}$$



What is the uncertainty Δx?

- Measure x many times: $-0.81a$, $-0.1a$, $0.3a$, . . .

[Aside: roll die(?) N times: N_1 1's, N_2 2's, . . .

avg # on face $= \frac{N_1}{N}\cdot 1 + \frac{N_2}{N}\cdot 2 + \cdots + \frac{N_6}{N}\cdot 6$

prob. of getting 6

$= \sum \mathbb{P}(\text{getting } n)\cdot n$

avg value of x

$$\langle x\rangle = \sum_{\text{bin } x} x\cdot \mathbb{P}(\text{bin } x) = \sum_{\text{bin } x} x\cdot \Delta x\, |\Psi(x)|^2 = \int_{-a}^{a} dx\, |\Psi|^2 \cdot x$$

$\langle x\rangle = 0$ (equally likely to be at $+x$ & $-x$) $= \left(\int_0^a dx + \int_{-a}^0 dx\right)|\Psi|^2$

equal in mag. opp sign.

$$\langle x^2\rangle = \int dx\, |\Psi(x)|^2 x^2 = \frac{a^2}{10}$$

4 How do we determine momentum in quantum mechanics?

Claim: $\langle p \rangle = \int_{-\infty}^{\infty} dx \, \Psi^* \left(-i\hbar \frac{d\Psi}{dx}\right)$

$\Psi = \Psi_1 + i\Psi_2$

$\Psi^* = \Psi_1 - i\Psi_2$

(flip $i \to -i$)

Intuition: $\Psi(x) = e^{ikx}$ (NOT NORMALIZED)

$\langle p \rangle = \int_{-\infty}^{\infty} dx \, e^{-ikx} \left(-i\hbar \cdot ik e^{ikx}\right) = \hbar k \underbrace{\int_{-\infty}^{\infty} dx \, \cancel{e^{-ikx}} \, \cancel{e^{ikx}}}_{= \int_{-\infty}^{\infty} dx \, |\Psi|^2}$ $= 1$

$p = \hbar \cdot k$

$\langle p^2 \rangle = \int_{-\infty}^{\infty} dx \, \Psi^* \left(-i\hbar \frac{d}{dx}\right)\left(-i\hbar \frac{d}{dx}\right) \Psi$

$= -\hbar^2 \int_{-\infty}^{\infty} dx \, \Psi^* \frac{d^2\Psi}{dx^2} = \cancel{-\hbar^2 \Psi^* \frac{d\Psi}{dx}\Big|_{-\infty}^{\infty}} + \int_{-\infty}^{\infty} dx \, \hbar^2 \frac{d\Psi^*}{dx} \frac{d\Psi}{dx}$

0

$\langle p^2 \rangle = \hbar^2 \int_{-\infty}^{\infty} dx \left|\frac{d\Psi}{dx}\right|^2$

$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$

5 Consider the wave function

$$\Psi(x) = \begin{cases} \overset{\text{real}}{A}(\overset{\text{real}}{a} - |x|) & |x| \leq a \\ 0 & x > a \end{cases} . \qquad [\Psi^* = \Psi]$$

What is the uncertainty Δp?

$$\langle p \rangle = \int_{-\infty}^{\infty} -i\hbar \, \Psi^* \frac{d\Psi}{dx} = -i\hbar \int_{-\infty}^{\infty} dx \, \underbrace{\Psi \frac{d\Psi}{dx}}_{\frac{1}{2}\frac{d}{dx}(\Psi^2)} = -\frac{i\hbar}{2} \Psi^2 \Big|_{-\infty}^{\infty} = 0$$

$$\frac{d\Psi}{dx} = \begin{cases} -A & a > x > 0 \\ A & 0 > x > -a \\ 0 & |x| > a \end{cases}$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} dx \, \hbar^2 \left| \frac{d\Psi}{dx} \right|^2$$

$$= \int_{-a}^{a} dx \, \hbar^2 A^2 = 2a \cdot \hbar^2 A^2 = 3\frac{\hbar^2}{a^2}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{3\frac{\hbar^2}{a^2}} = \sqrt{3}\frac{\hbar}{a}$$

$$\Delta x \Delta p = \left(\frac{a}{\sqrt{10}}\right)\left(\sqrt{3}\frac{\hbar}{a}\right) = \sqrt{\frac{3}{10}}\hbar \approx 0.55\hbar \geq \frac{\hbar}{2}.$$