PHYS 2170

General Physics 3 for Majors

Fall 2021

Lecture 31

The Schrödinger equation

November 8

1 Argue for the Schrödinger equation which governs quantum waves.

$\Psi(x,t)$ = particle's quantum wave function

interpret: upon measuring x

$\mathbb{P}(a \le x \le b) = \int_a^b dx\, |\Psi|^2$

prob.

$E = hf = \hbar\omega \qquad \hbar = \frac{h}{2\pi}$

$p = \frac{h}{\lambda} = \hbar k$

deduce Ψ equation:

$\Psi = e^{ikx - i\omega t}$

$E = \frac{1}{2}mv^2$ (non-rel)

$p = mv$

$E = \frac{p^2}{2m}$

$\hbar\omega = \frac{(\hbar k)^2}{2m}$

$$\boxed{\hbar\omega\Psi = \frac{\hbar^2 k^2}{2m}\Psi}$$

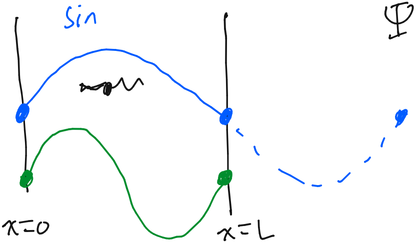
$-i\omega e^{ikx-i\omega t} = \frac{\partial}{\partial t} e^{ikx-i\omega t}$

$ik\, e^{ikx-i\omega t} = \frac{\partial}{\partial x} e^{ikx-i\omega t}$

$i\hbar(-i\omega)\Psi = -\frac{\hbar^2}{2m}(ik)^2\Psi$

$$\boxed{i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}}$$

2 Solve the Schrödinger equation in a box of length L by making an intuitive argument.



$$\Psi(0) = \Psi(L) = 0$$

$$\lambda_1 = 2L$$

In general... $\lambda_n = \frac{2L}{n}$ $(n = 1, 2, 3, \ldots)$

$E_1 > 0$ follows from Heis UP

$$\Delta x \sim L$$

$$\Delta p \gtrsim \hbar/2L$$

$$E \sim \frac{\Delta p^2}{2m}$$

Energy of longest wavelength mode?

$$E_1 = \hbar\omega_1 = \hbar\omega(k_1)$$

$$= \hbar\omega\left(\frac{\pi}{L}\right) = \frac{\hbar^2(\pi/L)^2}{2m}$$

In general: $\boxed{E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2}$

3 Solve the Schrödinger equation in a box of length L by solving the differential equations directly.

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}$$

Look for: $\Psi(x,t) = \Psi(x)\, e^{-i\omega t}$

$[\omega = E/\hbar]$

$$\underbrace{\hbar\omega}_{E}\,\Psi = -\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2}$$

$$\frac{d^2\Psi(x)}{dx^2} = -\frac{2mE}{\hbar^2}\Psi(x)$$

Define $k^2 = \dfrac{2mE}{\hbar^2}$

Try: $\Psi = e^{bx}$

$$\frac{d^2\Psi}{dx^2} = b^2\Psi = -k^2\Psi; \quad b = \pm ik$$

$$\Psi(x) = A_1 e^{ikx} + A_2 e^{-ikx}$$

Boundary conds:

$$\Psi(0) = 0 = A_1 + A_2$$

$$\Psi(L) = 0 = A_1 e^{ikL} - A_1 e^{-ikL}$$

$$= \underbrace{2iA_1}_{C} \sin(kL) = 0$$

So $kL = n\pi$ $(n = 1, 2, 3 \ldots)$

or $k = \dfrac{n\pi}{L}$

Deduce:

$$\Psi_n = C \sin\left(\frac{n\pi x}{L}\right) e^{-iE_n t/\hbar}$$

$$E_n = \hbar^2/2m\; k_n^2 = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2.$$

4 Normalize the allowed quantum states in the box.

Normalization: $$1 = \int_{-\infty}^{\infty} dx \, |\Psi|^2 = \int_0^L dx \, |\Psi_n|^2$$

in box

$$= \int_0^L C^2 \sin^2\left(\frac{n\pi x}{L}\right) \left| e^{-iE_n t/\hbar} \right|^2 dx$$

$\to 1$

$|e^{i\theta}| = |\cos\theta|^2 + |\sin\theta|^2$

$$\sin^2(z) = \frac{1}{2}(1 - \cos(2z))$$

$$1 = C^2 \int_0^L dx \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi x}{L}\right)\right] = C^2 \left[\frac{x}{2} - \frac{L}{n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]\Bigg|_0^L$$

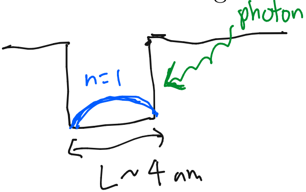
$$= C^2 \cdot \left[\frac{L}{2} - 0 - \frac{L}{n\pi}\underbrace{\sin(2n\pi)}_{0} + \frac{L}{n\pi}\underbrace{\sin(0)}_{0}\right] = C^2 \frac{L}{2}$$

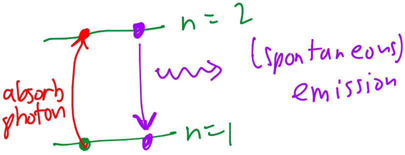
normalized:

Thus $C = \sqrt{\frac{2}{L}}$; $\Psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\left[e^{-iE_n t/\hbar}\right]$

5 In a quantum dot of length $L \approx 4$ nm sits an electron of mass $m \approx 10^{-30}$ kg. What is the longest wavelength of light it might absorb?

n=3





Allowed energies:

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2$$

$$= n^2 \cdot \left(3\times10^{-21}\ \text{J}\right)$$

$$E_1 + E_{ph} = E_2$$

$$E_{ph} = E_2 - E_1$$
$$= 3E_1$$

$$E_{ph} = 9\times10^{-21}\ \text{J}$$
$$= h f_{ph} = h\frac{c}{\lambda_{ph}}$$

$$[c = 3\times10^{8}\ \tfrac{m}{s}]$$

$$\lambda_{ph} = 2\times10^{-5}\ m$$

$$\boxed{\begin{aligned} c &= f\lambda \\ E &= cp \end{aligned}}$$