PHYS 2170

General Physics 3 for Majors

Fall 2021

Lecture 32

Quantum tunneling

November 10

1 Argue for the Schrödinger equation in an external potential.

Last time: $\Psi = e^{ikx - i\omega t}$ $\quad k = p/\hbar \quad \omega = E/\hbar$

For non-relativistic:

$$-i\omega \Psi = \frac{\partial \Psi}{\partial t}$$

(kinetic) $\quad E\Psi = \frac{p^2}{2m}\Psi$

$$\hbar\omega \Psi = \frac{(\hbar k)^2}{2m}\Psi$$

"free particle"

$$\boxed{i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}}$$

Potential energy?

$$E\Psi = \left(\frac{p^2}{2m} + U\right)\Psi$$

Schrödinger's Equation:

$$\boxed{i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + U(x)\Psi(x,t)}$$

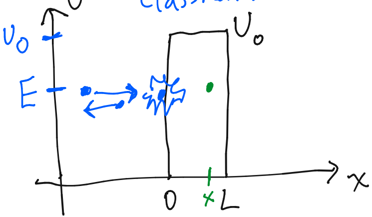
U(x) breaks solvability

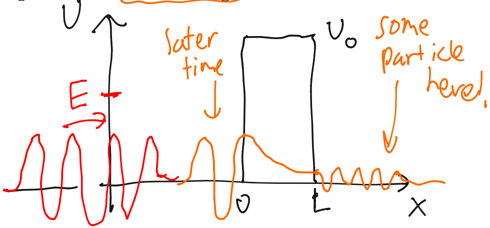
strategy: guess $\Psi(x,t) = \underbrace{e^{-iEt/\hbar}}_{e^{-i\omega t}}\Psi(x)$

$$E\Psi(x) = -\hbar^2/2m \frac{d^2\Psi}{dx^2} + U(x)\Psi(x)$$

2 Describe the phenomenon of quantum tunneling.

classical:





conservation of energy:

$$E = K(p) + U(x)$$

total

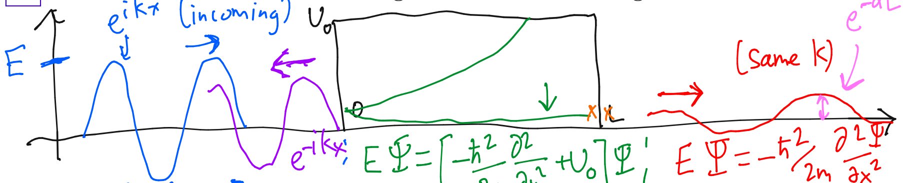
$$\text{“}\frac{p^2}{2m} \geq 0$$

Goal: estimate fraction of wave packet on right / probability of tunneling.

$$E = K + U(x) = K + U_0$$

$$K = E - U_0 < 0.$$

$\boxed{3}$ Stitch the wave function together in different regions.



$$E\Psi = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U\right]\Psi$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + 0.$$

$\Psi = e^{bx}$: $\quad b = \pm i\frac{\sqrt{2mE}}{\hbar}$

$$= \pm ik$$

incident

$$\Psi = A_1 e^{ikx}$$

$$+ A_2 e^{-ikx}$$

reflected wave

$$E\Psi = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U_0\right]\Psi$$

$$\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = (U_0 - E)\Psi$$

$\Psi = e^{bx}$:

$$\frac{\hbar^2}{2m}\cdot b^2 = U_0 - E$$

Define $\alpha = \frac{\sqrt{2m(U_0-E)}}{\hbar}$

$$b = \pm\alpha$$

$$\Psi = A_3 e^{-\alpha x} \cancel{+ A_4 e^{+\alpha x}}$$

$$E\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$$

$$\Psi = A_5 e^{ikx} \cancel{+ A_6 e^{-ikx}}$$

Need: Ψ continuous $[\Psi(0), \Psi(L)]$

$\frac{\partial\Psi}{\partial x}$ continuous

Intuition:

$$A_5 \sim A_3 e^{-\alpha L} \sim A_4 e^{\alpha L}$$

$$A_1 + A_2 \sim A_3 + A_4$$

$$\sim A_3(1 + \cancel{e^{-2\alpha L}})$$

$$A_5 \sim A_1 e^{-\alpha L}$$

$\boxed{4}$ Consider an electron of mass $m \approx 10^{-30}$ kg approaching the barrier

$\hbar \approx 10^{-34}$ J·s

$$U(x) = \begin{cases} 0 & x < 0 \text{ or } x > L \\ U_0 & 0 < x < L \end{cases},$$

with $U_0 = 2 \times 10^{-19}$ J. If it has kinetic energy $E = \frac{1}{2}U_0$, what is the probability it tunnels through if $L = 4$ nm?

Probability of tunneling:

$$\frac{|A_5|^2}{|A_1|^2} \sim e^{-2\alpha L}$$

$$\alpha = \frac{\sqrt{2m(U_0 - E)}}{\hbar}$$

For these numbers?

$$\alpha \approx \frac{\sqrt{2m \cdot \frac{1}{2} U_0}}{\hbar} = \frac{\sqrt{mU_0}}{\hbar}$$

$$\sim 4 \times 10^9 \text{ m}^{-1} \sim 4 \cdot \frac{1}{\text{nm}}$$

Probability $\sim 10^{-16}$