PHYS 2170

General Physics 3 for Majors

Fall 2021

Lecture 33

The Schrödinger equation in higher dimensions

November 12

1 What is the Schrödinger equation for a free particle in higher dimensions?

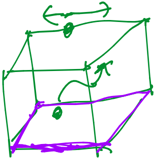
Recall: $E\Psi = \frac{p_{\cancel{x}}^2}{2m}\Psi$ $\qquad E = \hbar\omega \qquad p = \hbar k$

$e^{i\cdots} \qquad \omega = i\frac{\partial}{\partial t}[e^{ikx-i\omega t}]$

$$\boxed{k = -i\frac{\partial}{\partial x}}$$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} \cancel{+U(x)\Psi}$$

2d: $E\Psi = -\frac{\hbar^2}{2m}\left[\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\right]$



$$E = \frac{|\vec{p}|^2}{2m} = \frac{p_x^2 + p_y^2 + p_z^2}{2m}$$

$p_x = -i\hbar\frac{\partial}{\partial x}, \quad p_y = -i\hbar\frac{\partial}{\partial y}, \quad p_z = -i\hbar\frac{\partial}{\partial z}$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right]$$

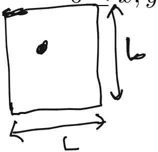
Solve $\Psi(x,y,z,t)$

$\Psi = e^{-i\omega t}\Psi(x,y,z)$

(time-indep.)

$$E\Psi = -\frac{\hbar^2}{2m}\left[\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right]$$

2 What are the energy levels of a particle confined to the square $0 \le x, y \le L$? Begin to solve the problem by separation of variables.



$$E\Psi = -\frac{\hbar^2}{2m}\left[\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\right]$$

$$\Psi(x,y) = X(x)\cdot Y(y)$$

[separation of variables]

$$E\cdot XY = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2}(XY) + \frac{\partial^2}{\partial y^2}(XY)\right]$$

$$\frac{\partial^2}{\partial x^2}\left[X(x)\,Y(y)\right] = \frac{\partial}{\partial x}\left[\frac{dX(x)}{dx}Y(y)\right]$$

$$= Y(y)\frac{d^2X}{dx^2}$$

$$E\cdot XY = -\frac{\hbar^2}{2m}\left[Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2}\right]$$

Divide by XY:

$$E = -\frac{\hbar^2}{2m}\left[\underbrace{\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}}_{\text{depend on } x} + \underbrace{\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}}_{\text{depend on } y}\right]$$

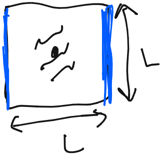
constant

const. = const + const.

E_x E_y

$$\boxed{\begin{aligned} E_x &= -\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2} \\ E_y &= -\frac{\hbar^2}{2m}\frac{1}{Y}\frac{d^2Y}{dy^2} \\ E &= E_x + E_y \end{aligned}}$$

3 What are the energy levels of a particle confined to the square $0 \leq x, y \leq L$?



$\frac{dy}{dx} = cxy$

Boundary conditions: $\Psi = 0$ at boundaries

$X(0) = X(L) = 0$

$\boxed{k_x = \frac{\sqrt{2mE_x}}{\hbar}}$ $\qquad \frac{d^2X}{dx^2} = -k_x^2 X$

$e^{i0} = 1$

$X(x) = A_1 e^{ik_x x} + A_2 e^{-ik_x x}$

Need to solve:

$$\boxed{-\frac{2mE_x}{\hbar^2} X(x) = \frac{d^2X}{dx^2}}$$

$$-\frac{2mE_y}{\hbar^2} Y(y) = \frac{d^2Y}{dy^2}$$

$X(0) = 0 = A_1 + A_2$, so $A_2 = -A_1$

$X(L) = 0 = A_1(e^{ik_x L} - e^{-ik_x L})$

$0 = \sin(k_x L)$, or $k_x = \frac{n_x \pi}{L}$

$n_x = 1, 2, 3, \ldots$

Same math for $Y(y)$:

$Y(y) = \sin(k_y y)$

$k_y = \frac{n_y \pi}{L}$, $n_y = 1, 2, 3, \ldots$

$$E = E_x + E_y = \frac{(\hbar k_x)^2}{2m} + \frac{(\hbar k_y)^2}{2m}$$

$$= \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 [n_x^2 + n_y^2]$$

normalization

$$\Psi(x, y) = A \cdot X(x) Y(y) = A \cdot \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right)$$

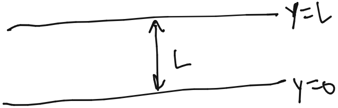
4 What are the 2 lowest energy levels of a particle confined to the square $0 \le x, y \le L$?

$$E = \frac{\pi^2 \hbar^2}{2mL^2}\left[n_x^2 + n_y^2\right] \qquad n_x, n_y = 1, 2, 3, \ldots$$

Lowest energy: $n_x = n_y = 1$

Next lowest? $\left.\begin{array}{c} n_x = 2 \text{ and } n_y = 1 \\ \underline{\text{OR}} \\ n_y = 2 \text{ and } n_x = 1 \end{array}\right\}$ multiple states at same energy: <u>degenerate</u>

5 What are the energy levels of a particle confined to the "strip" $-\infty < x < \infty$, $0 \le y \le L$?



$$E = -\frac{\hbar^2}{2m}\left[\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}\right]$$

Try: $\Psi = X(x)\,Y(y)$

$\downarrow$ Same manip.

$$E = E_x + E_y$$

$$-\frac{2mE_x}{\hbar^2} X(x) = \frac{d^2X}{dx^2}$$

$$-\frac{2mE_y}{\hbar^2} Y(y) = \frac{d^2Y}{dy^2}$$

Y: $Y(y) = \sin\left(\frac{n_y \pi}{L} y\right)$ $\quad n_y = 1, 2, 3, \ldots$

$$E_y = \frac{\hbar^2}{2m}\left(\frac{n_y \pi}{L}\right)^2$$

X: $X(x) = e^{ikx}$ $\quad$ k real: $-\infty < k < \infty$.

$$E_x = \frac{(\hbar k)^2}{2m}$$

Energy levels:

$$E = \frac{\hbar^2}{2m}\left[\underset{\uparrow \; x}{k^2} + \underset{\uparrow \; y}{\left(\frac{n_y \pi}{L}\right)^2}\right]$$

discrete (y)

continuous (x)