PHYS 2170

General Physics 3 for Majors

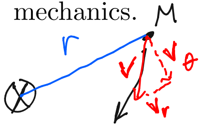
Fall 2021

Lecture 35

Angular momentum in two dimensions

November 17

1 Describe the central force problem in two dimensions in classical mechanics.



kinetic + potential:

$$E = \frac{1}{2}Mv^2 + U(r)$$

$\cancel{\vec{F} = m\vec{a}}$

Angular momentum conserved:

$$L_z = Mrv_\theta \qquad v_\theta = \frac{L_z}{Mr}$$

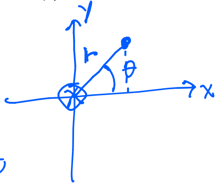
$$E = \frac{1}{2}M(\overbrace{v_r^2 + v_\theta^2}^{|\vec{v}|^2}) + U = \frac{1}{2}Mv_r^2 + \underbrace{\boxed{\frac{L_z^2}{2Mr^2} + U(r)}}_{U_{eff}(r)}$$

effective pot. energy

$\boxed{2}$ Repeat the calculation in quantum mechanics. $\left[\Psi(x,y,t) = e^{-iEt/\hbar}\,\Psi(x,y)\right.$

$$\underbrace{-\frac{\hbar^2}{2M}\left[\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}\right]}_{\nabla^2\Psi} + U(r)\Psi = E\Psi$$

polar coords: $x = r\cos\theta \qquad y = r\sin\theta$



$$-\frac{\hbar^2}{2M}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}\right] + U(r)\Psi = E\Psi$$

$E = $ const.

Separation of variables: $\Psi(r,\theta) = R(r)Y(\theta)$:

$$-\frac{\hbar^2}{2M}\left[\frac{1}{r}\left[\frac{d}{dr}r\left(\frac{dR}{dr}\right)\right]Y(\theta)+\frac{1}{r^2}R(r)\frac{d^2Y}{d\theta^2}\right] + U(r)RY = E\cdot RY$$

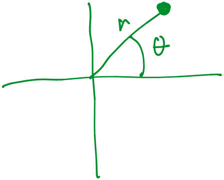
divide by RY

$$-\frac{\hbar^2}{2M}\left[\underbrace{\frac{r}{\cancel{2}R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)}_{r\text{ only}}+\underbrace{\cancel{\frac{1}{r^2}}\frac{1}{Y}\frac{d^2Y}{d\theta^2}}_{\theta\text{-dep.}}\right] + \underbrace{U(r)r^2}_{r\text{ only}} = \underbrace{Er^2}_{\text{const.}}$$

$$\frac{1}{Y}\left(-\frac{\hbar^2}{2M}\right)\frac{d^2Y}{d\theta^2} = \text{const.} \rightarrow \frac{\hbar^2}{2M}\alpha^2$$

3 Explain why angular momentum L_z is quantized.

$$\frac{1}{Y}\frac{d^2Y}{d\theta^2} = \text{const.} \quad ; \quad \frac{d^2Y}{d\theta^2} = \underbrace{-\alpha^2}_{\text{const.}} Y$$



Solutions: $Y(\theta) = A_1 e^{i\alpha\theta} + A_2 e^{-i\alpha\theta}$

constants

θ is an angular coordinate. θ is not uniquely defined: θ equivalent to $\theta + 2\pi$

$Y(\theta) = e^{i\theta \cdot m}$ ← $\underline{Y(\theta) = Y(\theta + 2\pi)}$

$$e^{\pm i\alpha\theta} = e^{\pm i\alpha(\theta+2\pi)} = e^{\pm i\alpha\theta}\underbrace{e^{\pm i\alpha 2\pi}}$$

$\cos(2\pi\alpha) + i\sin(2\pi\alpha) = 1 + i0 = 1$

(classical): $\alpha = 0, \pm 1, \pm 2, \ldots$ (any integer)

$$ER = -\frac{\hbar^2}{2M}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{\hbar^2}{2M}\frac{m^2}{r^2}R + U(r)R$$

nearly 1d: $U_{eff} = U + \frac{(m\hbar)^2}{2Mr^2}$ $\quad \frac{L_z^2}{2Mr^2}$

$\boxed{L_z = m\hbar}$ $\underline{L_z \text{ quantized}}$

4 Discuss the general spectrum of a quantum system with rotational invariance.

- For each m, different ODE for $R(r)$:

$$-\frac{\hbar^2}{2M}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(V + \frac{\hbar^2 m^2}{2Mr^2}\right)R(r) = E R(r)$$

Solve:

set of allowed energies $E_{n,m}$ $(n = 1, 2, 3, \ldots)$

$E_{1,m} < E_{2,m} < \cdots$

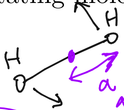
- typically differ for each m

- Degenerate energy?

$$E_{n,m} = E_{n,-m}.$$

5 A rotating molecule (in two dimensions) has ~~Hamiltonian~~ energy levels

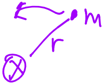
H_2



$m \leftarrow$ proton $\sim 2\times10^{-27}$ kg

$$\cancel{H}\,E = \frac{L_z^2}{2I}.$$

$\uparrow$ chemical bond $\approx 10^{-10}$ m



$I = mr^2$

Estimate I. If photon emission can only change L_z by $\pm\hbar$, what are the photons that can be emitted or absorbed?

$I \sim m\,a^2 \sim 2\times10^{-47}$ kg·m^2

$$E = \frac{L_z^2}{2I}.$$

Allowed energies are: $[L_z = m\hbar]$

$$E = \frac{(m\hbar)^2}{2I}$$

Absorb/emit photon: $m \to m\pm1$

$$E_{m+1} - E_m = \frac{\hbar^2}{2I}\left[(m+1)^2 - m^2\right] = \frac{\hbar^2}{2I}(2m+1)$$

$m = 0, 1, 2, \ldots$

Longest possible wavelength:

$m = 0 \leftrightarrow 1$

Smallest ΔE

$$\Delta E = \frac{hc}{\lambda}$$

$\lambda \sim 7\times10^{-4}$ m