PHYS 2170

General Physics 3 for Majors

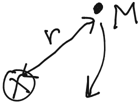
Fall 2021

Lecture 36

Angular momentum in three dimensions

November 19

1 Review the central force problem in two dimensions.



$$E = \frac{\vec{p}^{\,2}}{2M} + V(r)$$

In classical mech: $E = \frac{1}{2} M v_r^2 + \underbrace{\frac{L_z^2}{2Mr^2} + V(r)}_{\text{effective potential}}$

(effectively 1d)

In QM: look for states of energy E:

$$E\Psi = -\frac{\hbar^2}{2M}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\Psi)\right] + \left(V(r) + \frac{(m\hbar)^2}{2Mr^2}\right)\Psi$$

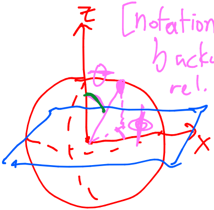
Angular momentum

$L_z = m\hbar$

$m = 0, \pm 1, \pm 2, \ldots$

2 State the solution to the central force problem in three dimensions in quantum mechanics.

In 3d:
$$-\frac{\hbar^2}{2M}\left[\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}\right]+U(r)\Psi=E\Psi$$



[notation backwards rel. to math]

$\downarrow$ spherical coords. (ugly!)

$$-\frac{\hbar^2}{2M}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\Psi)\right]+\left(U(r)+\frac{\vec{L}^2}{2Mr^2}\right)\Psi=E\Psi$$

$$\vec{L}^2=L_x^2+L_y^2+L_z^2$$

$$\vec{L}^2\Psi=\hbar^2\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}\right]$$

Try separation of variables: $\Psi=R(r)Y_1(\theta)Y_2(\phi)$

3 Describe the quantization of $\mathbf{L}^2$ and L_z in quantum mechanics.

$\Psi = R(r)\, Y_1(\theta)\, Y_2(\phi) \ldots$

- $Y_1(\theta) = f_{\ell m}(\theta)$

- $Y_2(\phi) = e^{im\phi}$ $(m = 0, \pm 1, \pm 2, \ldots)$

  $\hookrightarrow L_z = m\hbar$

$\ell = 0, 1, 2, \ldots$

$(\vec{L}^2 \geq L_z^2) \rightarrow \ell \geq |m|$

How many allowed states have $\ell = 3$?

- $3 \geq |m|$
- $m = -3, -2, -1, 0, 1, 2, 3$
- 7 possible choices

- In general, when $\ell > 0$, degeneracy of $2\ell + 1$

- $\vec{L}^2\, \Psi = \hbar^2 \ell(\ell+1)\, \Psi$
- total angular momentum is quantized

The effective radial potential:

$$U_{eff}(r) = U(r) + \frac{\hbar^2 \ell(\ell+1)}{2Mr^2}$$

depends only on ℓ

4 A rotating molecule (in ~~two~~ three dimensions) has energy levels

$$E\Psi = \frac{\mathbf{L}^2}{2I}\Psi \qquad \left[\text{rotational} \quad \frac{\vec{p}^{\,2}}{2M}\right]$$

What are the quantized energy levels? What is the degeneracy of the 3rd lowest energy level?

Allowed energy levels: $E_\ell = \frac{\hbar^2 \ell(\ell+1)}{2I}$ $\quad \ell = 0, 1, 2, \ldots$

3rd lowest energy: $\ell = 2$: $E_2 = \frac{\hbar^2 \cdot 2(2+1)}{2I} = 3\frac{\hbar^2}{I}$

degeneracy: 5

$L_z = -2\hbar, -\hbar, 0, \hbar, 2\hbar$

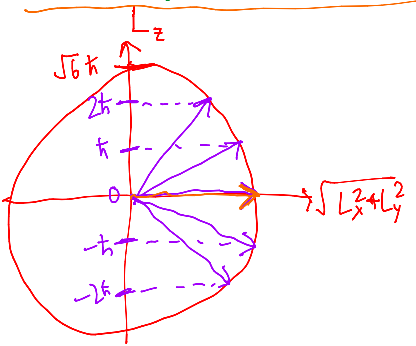
5 How is it possible to have $\mathbf{L}^2 > \max(L_z^2)$?

Take $\ell = 2$ states:

$$\vec{L}^2 = \hbar^2 \ell(\ell+1) = \hbar^2 \cdot 2 \cdot (2+1) = 6\hbar^2$$

Max value of L_z^2 : $(m=2)$: $(2\hbar)^2 = L_z^2 = 4\hbar^2$

$$\vec{L}^2 - L_z^2 = 6\hbar^2 - 4\hbar^2 = 2\hbar^2 = \hbar^2 + \hbar^2 .$$



max value of L_x, L_y, L_z should all be $2\hbar$ $(\ell\hbar)$

Heisenberg!:

$$\Delta L_x \Delta L_y \geq \frac{\hbar}{2} \langle L_z \rangle$$

Look at $m=2$:

$$\Delta L_x \Delta L_y \geq \frac{\hbar}{2} \cdot 2\hbar = \hbar^2$$

Guess: $\Delta L_x = \Delta L_y$ $\Delta L_x^2 = \Delta L_y^2 = \hbar^2 .$