PHYS 2170

General Physics 3 for Majors

Fall 2021

Lecture 37

Quantum solution of the hydrogen atom

November 29

1 Review the central force problem in three dimensions.

r $-e$ electron mass M

$+e$

$$E = \frac{\vec{p}^{\,2}}{2M} + U(r)$$

total energy   kinetic   potential

Quantum mechanics: $\rightarrow$ $$E\Psi = -\frac{\hbar^2}{2M}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\Psi + \underline{U(r)}\Psi$$

Go to spherical coords:

- angular / rotational
  - each Ψ can be chosen to have fixed
    $$\vec{L}^2 = \hbar^2 \ell(\ell+1) \quad [\ell = 0, 1, 2, \ldots]$$
  - and fixed
    $$L_z = m\hbar \rightarrow m = -\ell, -\ell+1, \ldots, \ell$$
    $$\boxed{|m| \le \ell}$$
    $2\ell+1$ possibilities

Radial dynamics:

$$U_{eff}(r) = \frac{\vec{L}^2}{2Mr^2} + U(r)$$

- Find allowed E, solve eqn for every $\underline{\ell}$:

Today:

$$U(r) = -\frac{e^2}{4\pi\epsilon_0 r} = -\frac{B}{r}$$

2 State the quantum solution to the hydrogen atom problem.

States: Ψ have 3 quantum numbers: $\Psi_{n\ell m}$

ℓ, m: angular momentum: $\ell = 0, 1, 2, \ldots$

$(\vec{L}^2)$ (L_z) $m = -\ell, -\ell+1, \ldots \ell$

n denotes energy level: remarkably

$$E_{n\ell} = \boxed{-\frac{MB^2}{2\hbar^2}} \cdot \frac{1}{n^2} \qquad (n = 1, 2, 3, \ldots) \quad \text{and} \quad \ell < n$$

$-E_R$ predicted by Bohr! [But Bohr said $L_z = n\hbar$]

E

(n, ℓ, m)

$-\frac{E_R}{9}$: $(3,0,0)$ $(3,1,-1)$ $(3,1,0)$ $(3,1,1)$ $(3,2,-2)$ $(3,2,-1)$ $(3,2,0)$ $(3,2,1)$ $(3,2,2)$

$-\frac{E_R}{4}$: $(2,0,0)$ $(2,1,-1)$ $(2,1,0)$ $(2,1,1)$

$-E_R$: $(1,0,0)$

ℓ n

$0 < 1$

3 Define and sketch atomic orbitals.

old jargon:

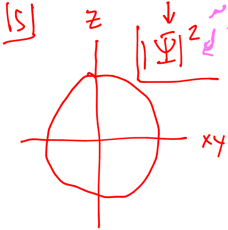
atomic orbital: $\Psi_{n\,l\,m}$

$2p$: $n = 2$, l (p: $l=1$), m (z)

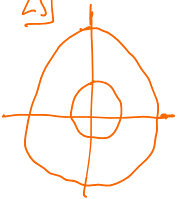
l	letter
0	s
1	p
2	d
3	f

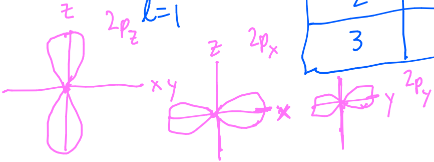
1s

$|\Psi|^2$ $\sim 1/a_B^3$



2s





E

$-\frac{E_R}{9}$ — (3,0,0) (3,1,-1) (3,1,0) (3,1,1) (3,2,-2) (3,2,-1) (3,2,0) (3,2,1) (3,2,2)

$-\frac{E_R}{4}$ — (2,0,0) (2,1,-1) (2,1,0) (2,1,1)

$2p_z$

(n,l,m)

$-E_R$ — (1,0,0)

1s

l n

$0<1$

$$\Psi_{2p_x} = \Psi_{2,1,1} + \Psi_{2,1,-1}$$

$$\Psi_{2p_y} = \Psi_{2,1,1} - \Psi_{2,1,-1}$$

4 What is the $\underbrace{\text{degeneracy}}$ of the n^{th} energy level?

how many values of ℓ and m are possible at fixed n.

First fix n & ℓ: how many m?

$m = -\ell, -\ell+1, \ldots, +\ell$ [$2\ell+1$ possible]

Vary ℓ: $\ell = 0, 1, \ldots, n-1$ [n possibilities]

$$\text{Degeneracy} = (\# \text{ w/ } \ell=0) + (\# \text{ w/ } \ell=1) + \cdots + (\# \text{ w/ } \ell=n-1)$$

$$= 1 + 3 + \cdots + \underbrace{2(n-1)+1}_{2n-1}$$

$$= \sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

$$= \cancel{2}\cdot\frac{n(n+1)}{\cancel{2}} - n$$

$$= n^2 + n - n = n^2$$

$\boxed{5}$ How many states have $n = 2$?

(last page) : $2^2 = 4$

$l=0 \quad m=0$

$l=1 \quad m=-1,0,1$

How many states have $m = 3$ if $l = 2$?

0

$|m| \leq l$.

How many states have $l = 4$ if $n = 5$?

9 :

$2l+1 = 2\cdot 4+1 = 9$.

How many states have $m = -4$ if $n = 5$?

1 :

$n=5$

$l=4$

$m=-4$

$\overset{5}{n} > \overset{4}{l} \geq \overset{4}{|m|}$