PHYS 2170

General Physics 3 for Majors

Fall 2021

Lecture 7

Relativistic energy and momentum

September 8

1 Review 4-velocity and relate it to ordinary velocity. Define 4-momentum.

4-velocity: (U_t, U_x, U_y, U_z)

$U_t = c\,\dfrac{dt}{d\tau}$

$U_x = \dfrac{dx}{d\tau}$ ← observer's coordinates

↖ proper time

4-momentum

$(p_t, p_x, p_y, p_z) = m \times (U_t, U_x, U_y, U_z)$

↑ mass of particle

$m =$ frame invariant.

If an object is at rest:

$(U_t, U_x, U_y, U_z) = (c, 0, 0, 0)$

Lorentz transform to frame moving at rel. velocity $\vec{v} = c\beta\hat{x}$

$U_t' = \gamma(U_t - \beta U_x)$

$U_x' = \gamma(U_x - \beta U_t)$

$U_y' = U_y$

$U_z' = U_z$

What is 4-velocity of a particle at $\vec{u} = c\beta\hat{x}$

[frame should rel. vel of $-c\beta\hat{x}$]

$U_t = \gamma c$

$U_x = \gamma c\beta = \gamma u$

$U_y = U_z = 0.$

$\gamma = \left[1 - u^2/c^2\right]^{-1/2}$

2 Taylor expand the components of 4-momentum to order v^2. What might they correspond to?

Moving at velocity $\frac{dx}{dt} = u_x, \frac{dy}{dt} = 0, \frac{dz}{dt} = 0$ $\quad |u_x| \ll c$:

4th order in $\frac{u_x}{c}$

$$p_x = mU_x = m\gamma u_x = \frac{mu_x}{\sqrt{1-u_x^2/c^2}} = mu_x\left[1 + \frac{1}{2}\frac{u_x^2}{c^2} + \cdots\right]$$

$$\simeq mu_x$$ [momentum!]

$$p_y = p_z = 0$$

$$p_t = mU_t = mc\gamma = mc\left[1 + \frac{1}{2}\frac{u_x^2}{c^2} + \cdots\right]$$

$$= mc + \frac{1}{2}m\frac{u_x^2}{c} + \cdots$$ energy??!

$\frac{\text{energy}}{c}$

$$E = cp_t \simeq mc^2 + \frac{1}{2}mu_x^2 + \cdots$$

rest energy

$E_{rest} = mc^2$

$$E = \gamma mc^2$$

$$\left[\gamma = \frac{1}{\sqrt{1-u_x^2/c^2}}\right]$$

Kinetic energy:

$E = (\gamma - 1)mc^2$

$\boxed{3}$ Show that all inertial frames agree on $(pc)^2 + (mc^2)^2 = E^2$.

Lorentz transform: (E, p_x) in frame S

frame S' moves at velocity $v_x = \beta c$

in frame S':

$$E' = \gamma(E - u_x p_x)$$

$$p_x' = \gamma\left(p_x - \frac{1}{c^2} u_x E\right)$$

$$p_t' = \gamma(p_t - \beta p_x)$$

$$p_x' = \gamma(p_x - \beta p_t)$$

$$p_t' = \frac{1}{c} E'$$

Claim: $E^2 = (p_x c)^2 + (mc^2)^2$

$$E = \gamma mc^2$$

$$p_x = \gamma m u_x = \gamma mc \beta_x$$

$$(\gamma mc^2)^2 \overset{?}{=} (\gamma mc^2 \beta_x)^2 + (mc^2)^2$$

$$\gamma^2 \overset{?}{=} \gamma^2 \beta_x^2 + 1$$

$$\gamma = \frac{1}{\sqrt{1-\beta_x^2}} \qquad \gamma^2(1-\beta_x^2) = 1 \checkmark$$

Frame invariant: $c^2(\Delta t)^2 - (\Delta x)^2$.

4 What is your energy at rest? Can you do anything useful with this energy?

$E_{rest} = mc^2$

$m \sim 100\,\mathrm{kg}$

$c \approx 3 \times 10^8\ \mathrm{m/s}$

$E_{rest} \sim 100(3 \times 10^8)^2$

$\sim 9 \times 10^{2+8\cdot 2} \sim 9 \times 10^{18} \sim 10^{19}\ \mathrm{J}$

nuclear warhead $\sim 5 \times 10^{15}$ J. $\longrightarrow$ 50 g $\rightarrow$ energy [photons]

This rest energy is not <u>usable</u> [w/o nuclear...];

$$E = \sqrt{(p_x c)^2 + (mc^2)^2} \geq mc^2.$$