**Physics 4410: Quantum Mechanics 2** **Fall 2020**

# Homework 1

**Due:** 11:59 PM MST, Tuesday, September 1. Submit your homework via Canvas.

5 points **Problem 1 (Quantum random number generator):** Suppose you are given a two level quantum system, such as the electronic state of an atom, and have the ability to measure the Pauli operators $\sigma^z$ and $\sigma^x$. Using the quantum mechanical principle of measurement, explain how (in principle) to build a perfect random number generator.[1]

10 points **Problem 2 (Quantum Zeno effect):** A three level quantum system has the Hamiltonian

$$H = a|1\rangle\langle 2| + a|2\rangle\langle 1| + b|2\rangle\langle 3| + b|3\rangle\langle 2| \tag{1}$$

(a) Assume $a, b > 0$. Find the eigenstates and eigenvectors of $H$. You may use `Mathematica`.

(b) Suppose that $b \gg a$. Describe the time evolution $|\psi(t)\rangle$ of the initial state $|\psi(0)\rangle = |1\rangle$. Show that in this limit the state $|\langle 1|\psi(t)\rangle| \approx 1$ at all times $t$. This is one version of the **quantum Zeno effect** – states such as $|1\rangle$ are effectively blocked from transitioning into rapidly mixing states, such as $|2\rangle$.

10 points **Problem 3 (Mesons):** Mesons are composite particles consisting of a quark-antiquark pair. They are among the zoo of elementary particles that are produced at particle colliders. Because of the nuclear strong force, there is a constant attractive force of strength $F$ pulling the quark and antiquark together.

(a) Assume each quark and antiquark are ultrarelativistic. Give a heuristic argument, based on the Heisenberg uncertainty principle, that if the quark and antiquark are separated by distance $L$, we can estimate their total energy as

$$E(L) = \frac{\hbar c}{2L} + FL. \tag{2}$$

In your argument, you can assume that the antiquark is immobile and the quark is mobile.

(b) Determine the separation $L$ of the quark and antiquark by minimizing the total energy of the pair.

(c) Use the relationship between rest mass and energy to estimate the mass $m$ of the meson.

(d) A typical meson has mass $m \approx 10^{-28}$ kg; $c \approx 3 \times 10^8$ m/s. Estimate the force $F$, and the size $L$ of the meson.

10 points **Problem 4 (Carbon nanotube):** Consider an electron bound to the surface of a small carbon nanotube – a thin cylinder of carbon atoms, of radius $R \approx 2 \times 10^{-10}$ m, which is one atom thick. Assume the nanotube has infinite length for simplicity, and that beyond sticking to the surface of the nanotube, the motion of the particle is free (in two dimensions).

[1] *Hint:* It suffices to be able to generate a string of "0" and "1", each which occurs with 50% probability, and each of which is uncorrelated with the previous bit. How should you assign these "0" and "1" in the quantum mechanical system?

(a) Find the eigenvalues and eigenstates of the mobile electron. Express your answer in terms of $R$, $m$, $\hbar$ and any discrete or continuous parameters you need to introduce to characterize the eigenstates.

(b) At room temperature, the typical energy of an electron is around $E_0 \sim 4 \times 10^{-21}$ J. Assuming that $m \approx 10^{-30}$ kg for convenience, argue that at this energy scale, the eigenstates of the Hamiltonian are equivalent to those of a free particle moving in one dimension.

10 points **Problem 5 (Flux quantization):** Superconductivity is a macroscopic manifestation of quantum mechanics, of which one property is that electrical currents flow without any resistance. In a superconductor, the mobile objects are Cooper pairs, bound states of two electrons, which can be thought of as particles of mass $m$ and charge $-2e$.

Consider a very narrow ring of radius $R$, made of a superconducting material, in a magnetic field $B$ pointing through the ring, as shown in Figure 1. The quantum Cooper pairs are constrained to move inside the ring, parameterized by angular coordinate $\theta$. The Hamiltonian for a Cooper pair is

$$H = \frac{1}{2m}\left(-\frac{\mathrm{i}\hbar}{R}\frac{\partial}{\partial\theta} + eBR\right)^2. \tag{3}$$

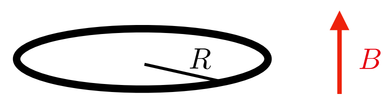


**Figure 1:** A magnetic field threading through a superconducting ring.

$m$ is the effective mass of the Cooper pairs, and can change from one material to the next.

(a) Find the eigenvalues and eigenstates of the effectively one dimensional Hamiltonian $H$.

(b) By comparing a superconductor with and without a Cooper pair excitation, argue that if Cooper pairs can freely form in a superconductor, then $E = 0$ must be an eigenstate of $H$. Conclude that the magnetic flux $\Phi_\mathrm{B} = \pi R^2 B$ through the loop must be quantized: $\Phi = n\Phi_0$ for integer $n$, and $\Phi_0$ a constant called the **magnetic flux quantum**. Argue that we can think of $\Phi_0$ as a fundamental constant: it is not sensitive to microscopic details of the geometry or even the superconducting material.

(c) A strong magnet found in a typical physics laboratory may be able to create $B \approx 1$ T. Given a loop with a very small $R$, and a loop with a very large $R$, which do you expect magnetic flux arising from this external field to be expelled from? Estimate the radius $R$ at which flux expulsion occurs.

10 points **Problem 6 (Möbius strip):** Consider the two-dimensional space $S = \{-\infty < x < \infty, 0 \le y \le L_y\}$: this is an infinite strip of width $L_y$.

(a) Find the eigenstates and eigenvalues of a free particle of mass $m$, constrained to move in this two-dimensional strip. Assume the boundary conditions $\psi(x, 0) = \psi(x, L_y) = 0$.

(b) Find all the eigenstates and eigenvalues that have the property that

$$\psi(x + L_x, y) = \psi(x, L_y - y). \tag{4}$$

(c) Argue that the solutions found in part (b) can also be thought of as the set of all solutions to the Schrödinger equation on a peculiar space, which we take by starting with $S$ and identifying (think gluing together) every point $(x, y)$ with the point $(x + L_x, L_y - y)$. Then, argue that this space is, loosely speaking, a **Möbius strip**.[2] (Mathematically speaking, the space that we obtain by identifying points in $S$ is said to be *topologically equivalent* to the Möbius strip.)

[2] *Hint:* If you are stuck, I would actually take a piece of graph paper and explicitly do this cutting and gluing/taping procedure. If you have never heard of the Möbius strip, just look it up online (e.g. YouTube).