Homework 10

Due: 11:59 PM, Tuesday, November 24. Submit your homework via Canvas.

Grading: 30 points required for full credit. 40 points are possible. You can score over 100%.

10 points **Problem 1:** A particle of mass m is in a one dimensional infinite square well: $0 \le x \le L$.

(a) Using the first order relativistic correction to the Hamiltonian

$$H = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \cdots \tag{1}$$

use first order perturbation theory to determine the eigenvalues of H to order c^{-2}.

(b) At what energy scale do you expect first order perturbation theory fails? Comment on the answer.

10 points **Problem 2 (Crystal field splitting):** Consider a hydrogen atom in a perturbing potential

$$H' = \frac{1}{2}\left(k_x x^2 + k_y y^2 + k_z z^2\right). \tag{2}$$

This is a toy model for how low-lying atomic energy levels can get distorted when atoms are found in a crystal lattice with a low symmetry group.

In this problem, you should *not* do any radial integrals. Instead, your job is only to qualitatively describe how the perturbation H' could split the high degeneracy of the hydrogen atom. You should begin by expressing x, y and z in terms of spherical coordinates. Then qualitatively sketch the evaluation of $\langle nlm|H'|nl'm'\rangle$, focusing on: (*1*) figuring out which radial integrals could be different; (*2*) which angular integrals might be non-zero. The hydrogen atom wave functions from Table 8.2 of McIntyre will be useful.

(a) What degeneracy could be lifted if $k_x = k_y = k_z$?

(b) Describe what happens in the $n = 1, 2$ sectors of the hydrogen atom if $k_x = k_y = -\frac{1}{2}k_z$: how many degenerate energy levels are lifted?

10 points **Problem 3:** Consider the Hamiltonian

$$H = \begin{pmatrix} 0 & \epsilon & \epsilon & 0 \\ \epsilon & 0 & 0 & 2\epsilon \\ \epsilon & 0 & 1 & 0 \\ 0 & 2\epsilon & 0 & 1-3\epsilon \end{pmatrix}. \tag{3}$$

Using degenerate perturbation theory, find the eigenvalues of H to second order in ϵ.

10 points **Problem 4 (Trapped ions):** In trapped ion crystals, including those studied at NIST in Boulder, the quantum spins on the ions are manipulated using "spin-dependent forces". Consider N spin-$\frac{1}{2}$ degrees of freedom (on the ions) coupled to a single one dimensional harmonic oscillator. Physically, this oscillator is a phonon mode, and corresponds to the collective vibration of the ion crystal.

A cartoon model for this system is as follows. The bare Hamiltonian is

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{4}$$

The Hilbert space of this system is then spanned by the following states: $|ns_1 s_2 \cdots s_N\rangle$, where $n = 0, 1, 2, \ldots$ denotes the oscillator's energy level, and $s_1, \ldots, s_N = \pm\frac{1}{2}$ denotes the z-component of the spins. The perturbation Hamiltonian corresponding to spin-dependent forces is

$$H' = -cx(S_{1z} + S_{2z} + \cdots + S_{Nz}). \tag{5}$$

The coefficient $c > 0$ is a constant.

(a) Explain why the states $|ns_1 \cdots s_N\rangle$ are all eigenvectors of H_0. What are their eigenvalues?[1]

(b) Show that

$$\langle n' s_1' \cdots s_N' | H' | n s_1 \cdots s_N \rangle = A_{n'n} \delta_{s_1 s_1'} \cdots \delta_{s_N s_N'}; \tag{6}$$

what is the form of the matrix $A_{n'n}$?[2]

(c) Use second order perturbation theory – being sufficiently careful when accounting for the degeneracy of H_0 – to evaluate the correction to the $n = 0$ energy levels. You should find that

$$E_0(s_1, \ldots, s_N) = K\,(s_1 + \cdots + s_N)^2\,. \tag{7}$$

Find the value of K in terms of m, ω, a and $\hbar$. Conclude that the low energy states in the Hilbert space are captured by an **effective Hamiltonian**

$$H_{\text{eff}} = \frac{4K}{\hbar^2}(S_{1z} + S_{2z} + \cdots + S_{Nz})^2. \tag{8}$$

[1] *Hint:* Recall how to deal with multi-particle Hamiltonians: see e.g. a problem on Homework 4.
[2] *Hint:* Use raising and lowering operators.