Homework 12

Due: 11:59 PM MST, December 6. Submit your homework via Canvas.

Grading: 35 points required for full credit. 35 points are possible.

Problem 1 (Chiral molecules): Consider a two dimensional molecule with two "inequivalent" L/R forms, as shown in Figure 1. If the world were two dimensional (namely we cannot rotate this molecule through the page), then these L and R molecules are classically distinguishable and not (without breaking bonds) convertible into one another by a rotation. We would then call this molecule chiral.

Quantum mechanically, however, can this finite quantum system really have two ground states? Let θ denote the relative angular displacement between the A and B atoms. Classically, there would be two ground states at $\theta = \pm 2\pi/3$. So we might crudely model this system with the quantum Hamiltonian

$$H = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\theta^2} + \frac{U}{2}\left(|\theta| - \frac{2\pi}{3}\right)^2. \qquad (1)$$

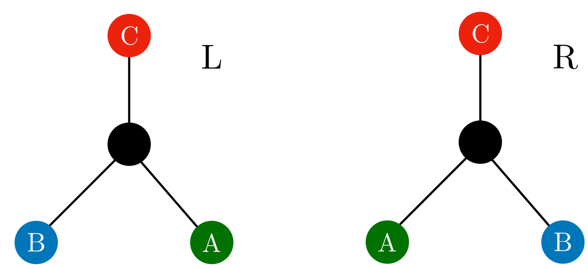

Figure 1: L/R forms of a chiral molecule in two dimensions.

Note that the potential has minima at $\theta = \pm 2\pi/3$.

5 points (a) Explain why $I \sim MR^2$, where R is the length of a chemical bond and M is the mass of a typical (e.g. carbon) atom. Then find the numerical value of I.

(b) Explain why U can (crudely) be estimated by the energy scale 10^{-18} J.

(c) Assuming that the wave function is trapped near $\theta = 2\pi/3$, estimate the ground state energy, and compare to U. Conclude that the wave function is effectively trapped at $\theta = 2\pi/3$, at least on short time scales.

10 points (d) Argue that if the particle starts at $\theta = 2\pi/3$, the time it will take to tunnel to $\theta = -2\pi/3$ can be *estimated* as

$$\tau = \sqrt{\frac{I}{U}}\exp\left[\frac{8\pi^2}{9\hbar}\sqrt{IU}\right]. \qquad (2)$$

(e) Find the numerical value of the tunneling time τ found in (2). Do you think it is reasonable for chemists and biologists to talk about molecular chirality – i.e., does a chiral molecule stay in the L (or R) state for a "long time"?

10 points **Problem 2 (Power law interactions):** Consider a particle of mass m in a one dimensional potential

$$V(x) = -C(|x| + x_0)^{-\alpha}. \tag{3}$$

Assume $C > 0$ and $\alpha > 0$. Use the Bohr-Sommerfeld approximation to argue that there are a finite number of bound states in this potential when $\alpha > 2$, and an infinite number of bound states for $0 < \alpha \leq 2$.

10 points **Problem 3:** Consider a particle of mass m in a 1d harmonic oscillator of frequency ω:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{4}$$

In the question that follows, calculus is not allowed!

(a) In classical mechanics, the (x, p) plane is often called **phase space**. Show that in phase space, the curves on which the Hamiltonian $H = E$ (here E is some constant) are ellipses. Sketch one such curve as accurately as you can, labeling axes and relevant scales properly.

(b) Use the geometric interpretation of the Bohr-Sommerfeld quantization condition to approximately quantize the harmonic oscillator. Compare to the exact answer.