# Homework 4

**Due:** 11:59 PM, Thursday, September 24. Submit your homework via Canvas.

**Problem 1:** Consider the Hamiltonian for two decoupled harmonic oscillators:

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{k}{2}x_1^2 + \frac{k}{2}x_2^2. \tag{1}$$

5 points (a) Define the particle exchange operator $P$ as follows:

$$x_1 = Px_2P, \tag{2a}$$
$$x_2 = Px_1P, \tag{2b}$$
$$p_1 = Pp_2P, \tag{2c}$$
$$p_2 = Pp_1P. \tag{2d}$$

Recall that $P^2 = 1$. Show that $[H, P] = 0$ (equivalently, $H = PHP$).[1]

5 points (b) Give an example of an operator $V$ that you could add to the Hamiltonian which would couple the two oscillators together (i.e., make $H + V$ not separable as a sum of operators acting on 1 plus operators acting on 2) while maintaining particle indistinguishability (i.e. $V = PVP$).

(c) Give an example of an operator $V$ that would break indistinguishability (i.e. $V \neq PVP$).

**Problem 2 (Product states):** Suppose that you have quantum system A, which can be found in one of $M$ possible states $|a_1\rangle, \ldots, |a_M\rangle$, and system B which can be found in one of $N$ possible states $|b_1\rangle, \ldots, |b_N\rangle$. Then the most general state of system A and B together is

$$|\psi_{\mathrm{AB}}\rangle = \sum_{i=1}^{M}\sum_{j=1}^{N} c_{i,j}|a_i\rangle \otimes |b_j\rangle, \tag{3}$$

where $|a_i\rangle \otimes |b_j\rangle$ is abstract notation for the simultaneous state of A being in state $|a_i\rangle$ and B being in state $|b_j\rangle$. McIntyre drops the explicit $\otimes$ symbol. There is a natural distributive property

$$(\alpha|a_1\rangle + \beta|a_2\rangle) \otimes |b_1\rangle = \alpha|a_1\rangle \otimes |b_1\rangle + \beta|a_2\rangle \otimes |b_1\rangle. \tag{4}$$

In this sense, $\otimes$ "multiplies" the Hilbert spaces of systems A and B!

Suppose that $\langle a_i|a_j\rangle = \delta_{ij}$ and $\langle b_i|b_j\rangle = \delta_{ij}$ – i.e. both bases above are orthonormal. Then we define

$$(\langle a_i| \otimes \langle b_k|)\,(|a_j\rangle \otimes |b_l\rangle) = \delta_{ij}\delta_{kl}. \tag{5}$$

In words, the bra of a product of kets is the product of bras. Coefficients are complex conjugated when going from ket to bra, just as usual.

[1] *Hint:* Consider the following chain of identities: $p_1^2 = p_1p_1 = p_1P^2p_1$.

5 points (a) For $n = 1, 2$, define the states

$$|\psi_n\rangle = \sum_{i=1}^{M} \psi_{n,i}|a_i\rangle, \qquad |\phi_n\rangle = \sum_{i=1}^{N} \phi_{n,i}|b_i\rangle. \tag{6}$$

By using the distributive property, show that

$$(\langle\psi_1| \otimes \langle\phi_1|)(|\psi_2\rangle \otimes |\phi_2\rangle) = \langle\psi_1|\psi_2\rangle\langle\phi_1|\phi_2\rangle. \tag{7}$$

5 points (b) Let $\mathcal{O}_\mathrm{A}$ be an operator acting on only system A, e.g.

$$\mathcal{O}_\mathrm{A}|a_i\rangle \otimes |b_k\rangle = \sum_{j=1}^{M} \mathcal{O}_{ji}|a_j\rangle \otimes |b_k\rangle. \tag{8}$$

In lecture, we said that $\mathcal{O}_\mathrm{A}$ was $\mathcal{O}_\mathrm{A}1_\mathrm{B}$, with 1 denoting the identity operation (doing nothing to the state of B). Either by explicit calculation, or crisp use of earlier identities, explain why

$$(\langle\psi_1| \otimes \langle\phi_1|)\mathcal{O}_\mathrm{A}(|\psi_2\rangle \otimes |\phi_2\rangle) = \langle\psi_1|\mathcal{O}_\mathrm{A}|\psi_2\rangle\langle\phi_1|\phi_2\rangle. \tag{9}$$

The state $|\psi_2\rangle \otimes |\phi_2\rangle$ is called a **product state** and is particularly easy to work with, because of the fact that inner products and simple expectation values "factorize" onto problems on systems A and B separately. You should memorize these main results of this problem, as they will prove essential in doing calculations in multi-particle systems.

**Problem 3 (Ferromagnetism):** Consider two non-interacting spin-$\frac{1}{2}$ electrons of mass $m$ in an infinite square well of length $L$. Let $|n_1 n_2\rangle$ denote the infinite square well energy eigenstates of electrons 1 and 2 respectively, and $|s_1 s_2\rangle$ denote the electron spin eigenstates (in the $z$-direction) of each particle.

5 points (a) Which of the following wave functions are allowed, given that electrons are fermions?

$$|a\rangle = \frac{|12\rangle - |21\rangle}{\sqrt{2}} \otimes \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}, \tag{10a}$$

$$|b\rangle = \frac{|12\rangle + |21\rangle}{\sqrt{2}} \otimes \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}, \tag{10b}$$

$$|c\rangle = |11\rangle \otimes \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}, \tag{10c}$$

$$|d\rangle = \frac{|12\rangle - |21\rangle}{\sqrt{2}} \otimes |--\rangle, \tag{10d}$$

$$|e\rangle = |11\rangle \otimes |--\rangle, \tag{10e}$$

$$|f\rangle = \left(\sqrt{\frac{1}{3}}|12\rangle - \sqrt{\frac{2}{3}}|21\rangle\right) \otimes |--\rangle. \tag{10f}$$

You should find that 3 out of 6 are acceptable wave functions.

5 points (b) For the acceptable wave functions, calculate $r$, defined as

$$r = \sqrt{\langle(x_1 - x_2)^2\rangle}, \tag{11}$$

with expectation value taken in the appropriate quantum state. Use `Mathematica` (if desired) to evaluate integrals. You should find that the three allowable wave functions have $r/L \approx 0.196, 0.256, 0.41$.[2]

[2] *Hint:* Read McIntyre Section 13.2 for a partial answer (although he also leaves integral evaluation to homework). Also, use the result of Problem 2 to get rid of the spin part of each wave function as soon as possible!

(c) For each of the 3 acceptable wave functions, determine the energy (call it $E_{\text{kin}}$) of the state in the infinite square well. Almost no calculation is needed.

5 points (d) In reality, the electrons will interact via Coulomb interactions. We can estimate the energy of a given wave function to be

$$E_{\text{tot}} \approx E_{\text{kin}} + \frac{e^2}{4\pi\epsilon_0 r}. \tag{12}$$

Using the mass and charge of the physical electron, along with the values of fundamental constants, determine the minimal value of $E_{\text{tot}}$ as a function of $L$, along with the wave function that minimized the energy. Show that above vs. below the length $L \approx 0.5$ nm, the "ground state wave function" which minimizes $E_{\text{tot}}$ changes.

(e) Iron is a **ferromagnet**, a phase of matter in which electronic spins spontaneously align with one another. Since the critical length scale above is comparable to atomic length scales, is it plausible that "exchange interactions" cause some materials to spontaneously magnetize? The effect is plausible so long as the energy gap between spin polarized and spin unpolarized states is at least as large as the thermal energy $E_{\text{thermal}} \sim 4 \times 10^{-21}$ J. At the critical $L$ above, evaluate $E_{\text{tot}}$; give a handwavy argument whether your model is consistent with room temperature ferromagnetism, or not, in iron.

**Problem 4 (Quarks):** Quantum chromodynamics (QCD) tells us that the ordinary matter we see around us, made up of protons and neutrons, is itself made up of more fundamental excitations called quarks and antiquarks. Note that quarks can be distinguished from antiquarks, but two quarks are indistiguishable fermionic particles, as are two antiquarks. A quark has 3 internal states, which we call "color states": red $|\mathrm{r}\rangle$, green $|\mathrm{g}\rangle$, and blue $|\mathrm{b}\rangle$. An antiquark is similar, but we denote the states with $|\bar{\mathrm{r}}\rangle$, $|\bar{\mathrm{g}}\rangle$ and $|\bar{\mathrm{b}}\rangle$ to emphasize that quarks and antiquarks are distinguishable particles.

5 points (a) A **baryon** is a particle made out of 3 quarks in the wave function

$$|\text{baryon}\rangle = \frac{|\mathrm{rgb}\rangle + |\mathrm{gbr}\rangle + |\mathrm{brg}\rangle - |\mathrm{grb}\rangle - |\mathrm{bgr}\rangle - |\mathrm{rbg}\rangle}{\sqrt{6}}. \tag{13}$$

Given that quarks are fermions, is that an acceptable wave function?

(b) A **meson** is a particle made out of a quark and an antiquark, in the wave function

$$|\text{meson}\rangle = \frac{|\mathrm{r}\bar{\mathrm{r}}\rangle + |\mathrm{g}\bar{\mathrm{g}}\rangle + |\mathrm{b}\bar{\mathrm{b}}\rangle}{\sqrt{3}}. \tag{14}$$

This wave function is not symmetric under the exchange of the two particles. Why is that allowed?

5 points (c) Argue that mesons are bosonic particles, while baryons are fermionic particles.[3]

(d) Now suppose there was a fourth color state, yellow $|\mathrm{y}\rangle$. The baryon wave function (13) would need to be modified so that it contains 4 quarks, one in each color state, suitably antisymmetrized. Would the baryon be a boson or a fermion?

[3] *Hint:* To exchange two mesons, we have to exchange a quark and an antiquark. What is the *net* contribution to the sign change of a multi-particle wave function from applying the appropriate number of particle exchange operations (one for each fundamental particle that needs to swap)? It might be useful to think of the schematic form of the wave function of 2 mesons: $|\mathrm{r}\bar{\mathrm{r}}\rangle \otimes |\mathrm{r}\bar{\mathrm{r}}\rangle + \cdots$. What would happen to this wave function if we swapped only the quarks? Once you understand the meson case, repeat the argument for baryons.

**Problem 5 (Pigments):** A typical pigment molecule has the structure sketched in Figure 1, and consists of a long chain of covalently bonded atoms which can be approximated as a one dimensional infinite square well. Suppose there are $N$ atoms in the long chain of atoms: then each bond contributes one "free" spin-1/2 electron, of mass $m$, which may move up and down the chain of bonds freely. If each bond has length $a$, when $N$ is large, we may thus approximate these electrons as moving in an infinite square well of width $L = Na$.

5 points (a) Using the Pauli exclusion principle and the energy levels for the particle in a box (i.e. infinite square well), describe which energy levels in the box are filled and which are empty in the ground state. Ignore electron-electron interactions.

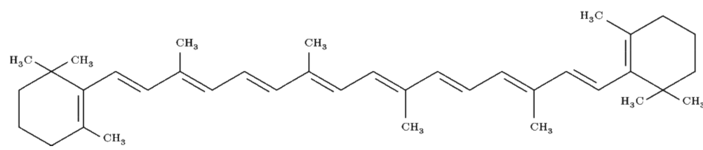


**Figure 1:** The β-carotene molecule is responsible for the orange color of carrots.

5 points (b) Now, suppose we send a photon of wavelength $\lambda$ at the pigment molecule. What is the largest value of $\lambda$ such that the photon can be absorbed by an electron in the pigment molecule? Assume $N > 1$. When the photon is absorbed, the electron must be able to jump to an unoccupied state in the box. You should find that when $N \gg 1$,

$$\lambda \approx \frac{4cma^2}{\pi\hbar} N. \tag{15}$$

(c) We might estimate that $N = 18$ for β-carotene, as depicted in Figure 1. Evaluate $\lambda$, given that $m \approx 9 \times 10^{-31}$ kg and $a \approx 10^{-10}$ m, and compare to the wavelength of orange light: 600 nm.