Homework 5

Due: 11:59 PM, Tuesday, October 6. Submit your homework via Canvas.

10 points **Problem 1 (White dwarf):** A **white dwarf** is a star in which the quantum mechanical "degeneracy pressure" arising from a Fermi gas of mobile electrons prevents gravitational collapse. In this problem, we consider a white dwarf to be a star of radius R, containing a density n of nucleons of mass m_{nuc}. Assume that every nucleon contributes one mobile electron of mass m to a Fermi gas. The gravitational potential energy of a white dwarf of N total nucleons can be approximated as

$$V_{\text{grav}} = -\frac{3GM^2N^2}{5R}. \tag{1}$$

(a) What is the number density of mobile electrons? Since electrons have spin-$\frac{1}{2}$, what is the corresponding energy density of the Fermi gas?[1] Conclude that the total energy of the white dwarf is given by

$$E_{\text{tot}} = -\frac{3GM^2N^2}{5R} + \left(\frac{3}{2\pi}\right)^{7/3} \frac{\pi^3\hbar^2N^{5/3}}{5mR^2} \tag{2}$$

(b) Find the radius R of the white dwarf by minimizing E_{tot}:

$$R = \left(\frac{9\pi}{4}\right)^{2/3} \frac{\hbar^2}{GmM^2N^{1/3}}. \tag{3}$$

(c) Numerically determine the radius of a white dwarf with a mass comparable to our sun, 10^{30} kg. Estimate that the nucleon mass $M \approx 10^{-27}$ kg and the electron mass $m \approx 10^{-30}$ kg in your calculation.

Problem 2 (Bloch oscillations): In Lecture 12, we found the following dispersion relation for electrons hopping on a lattice:

$$E(k) = \alpha - 2\beta\cos(ka). \tag{4}$$

5 points (a) Evaluate the group velocity as a function of k.

5 points (b) In an external force of strength F, we can heuristically think of a quantum wave packet as having a time dependent mean wave number:

$$\bar{k}(t) = \frac{Ft}{\hbar}. \tag{5}$$

The mean position of the wave packet can then be found by solving the differential equation

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = v_{\text{g}}(\bar{k}(t)). \tag{6}$$

Show that the position $\bar{x}$ of the wave packet oscillates in time. These are called **Bloch oscillations** and are a striking consequence of the discrete translation symmetry on the lattice.

[1] *Hint:* The answer is given in Lecture 11 – scan through the recorded lecture if you need to!

Problem 3: Consider the following modification of the model we studied in Lecture 14: for integer n',

$$H|2n'\rangle = \alpha|2n'\rangle - \zeta_1|2n'+1\rangle - \zeta_2|2n'-1\rangle, \tag{7a}$$
$$H|2n'+1\rangle = \alpha|2n'+1\rangle - \zeta_1|2n'\rangle - \zeta_2|2n'+2\rangle. \tag{7b}$$

Assume α is real, and $\zeta_1, \zeta_2 > 0$.

5 points (a) Show that $[H, T_2] = 0$. Conclude that we can look for eigenvectors of H of the form

$$|\psi\rangle = \sum_{n'=-\infty}^{\infty} \mathrm{e}^{2\mathrm{i}n'\theta} \left[c_1|2n'\rangle + c_2\mathrm{e}^{\mathrm{i}\theta}|2n'+1\rangle\right], \tag{8}$$

for unknown parameters c_1 and c_2.

5 points (b) Show that the eigenvalues of H are the eigenvalues of the following 2×2 matrix:

$$H(\theta) = \begin{pmatrix} \alpha & -\zeta_1\mathrm{e}^{\mathrm{i}\theta} - \zeta_2\mathrm{e}^{-\mathrm{i}\theta} \\ -\zeta_1\mathrm{e}^{-\mathrm{i}\theta} - \zeta_2\mathrm{e}^{\mathrm{i}\theta} & \alpha \end{pmatrix}. \tag{9}$$

What values of θ should we restrict to?

5 points (c) Find the eigenvalues of $H(\theta)$. Sketch the resulting band structure (eigenvalues of the Hamiltonian as a function of θ). Describe, as a function of Fermi energy E_F, whether this model describes a metal or an insulator.[2]

5 points BONUS **Problem 4 (Power law hopping):** Consider a quantum particle hopping between sites of a one-dimensional lattice according to the Hamiltonian

$$H = -\sum_{\substack{m,n=-\infty \\ m\neq n}}^{\infty} \frac{h}{|m-n|^\alpha}|n\rangle\langle m|. \tag{10}$$

(a) Let $T = |n+1\rangle\langle n|$ denote the translation operator. Show that $[H, T] = 0$.

(b) Show that the eigenvalues of H are given by

$$E(k) = -2h\sum_{n=1}^{\infty} \frac{\cos(nka)}{n^\alpha}, \quad |k| \le \pi a. \tag{11}$$

Here a is a variable with the units of length which represents the spacing between atoms in the lattice.

(c) Show that $E(0)$ is only finite if $\alpha > 1$. Hence we should restrict to this regime, or else the model is not physical.

(d) Assume $\alpha > 3$. Taylor expand $\cos(nka)$ to quadratic order in k. Then argue that as $k \to 0$,

$$E(k) \approx E(0) + \frac{\hbar^2 k^2}{2m_\mathrm{eff}} + \cdots \tag{12}$$

with $m_\mathrm{eff} < \infty$. Even though particles can hop arbitrarily large distances, low energy excitations behave as if they are nice and local!

[2] *Hint:* You can make a sketch as follows. What are the eigenvalues $E(\theta)$ at $\theta = 0$; what about $\theta = \pi/2$? Sketch two bands that connect these two points (for each band) together. Repeat for negative θ.

(e) For $\alpha < 3$, we can estimate the behavior of $E(k)$ as $k \to 0$ as follows: starting with (11), we write

$$E(k) - E(0) \approx -2h \int_1^\infty \mathrm{d}n \, \frac{\cos(nka) - 1}{|n|^\alpha}. \tag{13}$$

Justify this approximation. Next, do a change of variable in the integral to $z = nka$. Without analytically evaluating the integral, explain why as $|k| \to 0$,

$$E(k) \approx E(0) + 2h \times C_\alpha |k|^{\alpha-1}. \tag{14}$$

where C_α is a constant that depends on α (you do not need to compute it).

(f) Information propagates at a finite speed so long as the group velocity of waves is finite. Using the dispersion relation (14), show that for $\alpha > 2$, information travels at a finite speed.[3]

[1] M. C. Tran, C-F. Chen, A. Ehrenberg, A. Y. Guo, A. Deshpande, Y. Hong, Z-X. Gong, A. V. Gorshkov and A. Lucas. "Hierarchy of linear light cones with long-range interactions", *Physical Review* **X10** 031009 (2020), `arXiv:2001.11509`.

[3] For a rigorous demonstration of this result for more general Hamiltonians, see Theorem 9 of [1].