Homework 7

Due: 11:59 PM, Thursday, October 22. Submit your homework via Canvas.

10 points **Problem 1 (Quantum state transfer):** An important part of any future quantum computing hardware will be the ability to efficiently transfer a qubit from one degree of freedom to another. Consider two spin-$\frac{1}{2}$ degrees of freedom with Hamiltonian

$$H = A\mathbf{S}_1 \cdot \mathbf{S}_2. \tag{1}$$

Prepare the spins in the initial state

$$|\psi(0)\rangle = (a|{+}\rangle + b|{-}\rangle) \otimes \frac{|{+}\rangle + |{-}\rangle}{\sqrt{2}} = \frac{a|++\rangle + a|+-\rangle + b|-+\rangle + b|--\rangle}{\sqrt{2}} \tag{2}$$

A **quantum state transfer protocol** would send this state to

$$|\psi_{\mathrm{goal}}\rangle = \frac{|{+}\rangle + |{-}\rangle}{\sqrt{2}} \otimes (a|{+}\rangle + b|{-}\rangle) = \frac{a|++\rangle + b|+-\rangle + a|-+\rangle + b|--\rangle}{\sqrt{2}}. \tag{3}$$

Show that there is a time τ for which, evolving $|\psi(0)\rangle$ with Hamitonian (1), $|\psi(\tau)\rangle = \mathrm{e}^{\mathrm{i}\theta}|\psi_{\mathrm{goal}}\rangle$ for some real constant θ. Since we do not care about the overall phase of the wave function, we thus achieve perfect state transfer!

Problem 2 (Deuterium): The deuterium nucleus consists of a bound state of a proton and a neutron. In nuclear physics, it can be a useful *approximation* that the proton and neutron are two species $|\mathrm{p}\rangle$ and $|\mathrm{n}\rangle$ of an indistinguishable "nucleon" particle with 2 flavors. The approximate "nucleon" particle (which can be in either the proton or neutron state) is a spin-$\frac{1}{2}$ fermion.

5 points (a) The quantum state of deuterium can be approximated as follows:

$$|\psi\rangle = \frac{|\mathrm{pn}\rangle - |\mathrm{np}\rangle}{\sqrt{2}}|\psi_{\mathrm{spin}}\rangle. \tag{4}$$

Suppose that nucleon particle exchange symmetry was exact. Conclude that $|\psi_{\mathrm{spin}}\rangle$ must correspond to a spin triplet ($\mathbf{I}^2|\psi_{\mathrm{spin}}\rangle = 2\hbar^2|\psi_{\mathrm{spin}}\rangle$) state. Therefore, deuterium has spin 1.

5 points (b) Let $\mathbf{I}$ denote the total spin of the deuterium nucleus, and let $\mathbf{S}$ denote an electron spin. The hyperfine interaction of the deuterium atom is governed by the Hamiltonian

$$H = A\mathbf{S} \cdot \mathbf{I}. \tag{5}$$

Find the eigenvalues of H, along with their degeneracies.

Problem 3 (Baryons): The Standard Model of particle physics predicts a zoo of *a lot* of elementary particles. A subset of them are called **baryons**, which are made up of 3 indistinguishable spin-$\frac{1}{2}$ fermions called quarks. Hence, the baryons are fermions also.

In this problem, we'll focus only on two kinds of quarks. Consider the up quark (u) and the down quark (d) to be the "$+\frac{1}{2}$" and "$-\frac{1}{2}$" states of a fake "quark isospin" symmetry $\mathbf{J}$:

$$J_z|\mathrm{u}\rangle = +\frac{1}{2}|\mathrm{u}\rangle, \tag{6a}$$

$$J_z|\mathrm{d}\rangle = -\frac{1}{2}|\mathrm{d}\rangle. \tag{6b}$$

We're setting $\hbar = 1$ for this problem. We assume that $[J_x, J_y] = \mathrm{i}J_z$, etc., and so isospins can be added together just like ordinary angular momenta or spins. Note that quarks are charged particles under electromagnetism: the electric charge of the up quark is $2e/3$, and the electric charge of the down quark is $-e/3$.

The baryon wave function can schematically be written as follows:

$$|\text{baryon}\rangle = |\text{isospin and spin}\rangle|\text{color}\rangle \tag{7}$$

Note that the spin and isospin of all 3 quarks is contained within $|\text{isospin and spin}\rangle$; the color of all 3 quarks is contained within $|\text{color}\rangle$. In Homework 4 we've already seen that the color wave function of a baryon is antisymmetric under the exchange of any two fermions.[1] Therefore, $|\text{isospin and spin}\rangle$ should be a symmetric wave function under the exchange of any two quarks!

Now let's unpack $|\text{isospin and spin}\rangle$ a bit more carefully. Let's start by ignoring particle indistinguishability and describing what happens when we add together three spins of spin-$\frac{1}{2}$. (Denote the corresponding spin operators with $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$). Note that parts (a) to (d) are independent of our particle physics application.

5 points **(a)** Explain why $\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} = \frac{3}{2}\oplus\frac{1}{2}\oplus\frac{1}{2}$.[2]

5 points **(b)** Let $|\pm\rangle$ denote spin-$\frac{1}{2}$ states (eigenstates of S_{1z}, S_{2z} or S_{3z}). Argue that in the total spin $s = \frac{3}{2}$ sector, the highest z-spin state has to be

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = |+\rangle|+\rangle|+\rangle. \tag{8}$$

(c) Apply the total lowering operator $S_{1-} + S_{2-} + S_{3-}$ to this state three times. Normalizing the states, conclude that

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \frac{|+\rangle|+\rangle|-\rangle + |+\rangle|-\rangle|+\rangle + |-\rangle|+\rangle|+\rangle}{\sqrt{3}}, \tag{9a}$$

$$\left|\frac{3}{2}, -\frac{1}{2}\right\rangle = \frac{|+\rangle|-\rangle|-\rangle + |-\rangle|-\rangle|+\rangle + |-\rangle|+\rangle|-\rangle}{\sqrt{3}}, \tag{9b}$$

$$\left|\frac{3}{2}, -\frac{3}{2}\right\rangle = |-\rangle|-\rangle|-\rangle. \tag{9c}$$

[1] Why it is always the color wave function which is antisymmetric, as opposed to isospin or spin, is because there is a very large symmetry group for colors called SU(3), and the Slater determinant wave function we wrote is the unique one that is invariant under this!

[2] *Hint:* Start with $\frac{1}{2}\otimes\frac{1}{2} = 1\oplus 0$ as discussed in Lecture 18. What this means is that we were able to move from the $|m_1 m_2\rangle$ basis to the $|jm\rangle$ basis, with either $j = 1$ or $j = 0$. If we fix $j = 1$, then those three vectors form a spin 1 Hilbert space. What if we then multiply this vector space by spin-$\frac{1}{2}$ – isn't this just like the addition of two angular momenta? Finish the argument.

5 points (d) There are two sets of spin-$\frac{1}{2}$s in the total Hilbert space. Following part (a), argue that one set of spin-$\frac{1}{2}$ states takes the form

$$\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle_{\rm A} = \frac{|+\rangle|-\rangle - |-\rangle|+\rangle}{\sqrt{2}}|\pm\rangle. \tag{10}$$

Then use the fact that there are 3 total wave functions in the total $S_z = \pm\frac{1}{2}$ Hilbert space to conclude that the other spin-$\frac{1}{2}$ wave functions are:[3]

$$\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle_{\rm B} = \frac{2|\pm\rangle|\pm\rangle|\mp\rangle - |\pm\rangle|\mp\rangle|\pm\rangle - |\mp\rangle|\pm\rangle|\pm\rangle}{\sqrt{6}}. \tag{11}$$

5 points bonus (e) Now, let's try to build particle wave functions! Start by looking for particles of net spin $\frac{3}{2}$. Show that there is a set of 4 particles of physical spin $\frac{3}{2}$, corresponding to an isospin-$\frac{3}{2}$ "multiplet", with wave functions that we might schematically write as

$$|\text{isospin and spin}\rangle = \left|\frac{3}{2}, j_z\right\rangle_{\text{isospin}} \left|\frac{3}{2}, s_z\right\rangle_{\text{spin}} \tag{12}$$

Here we are using the constructions of parts (b) through (d) for both spin and isospin – explain why! Is this a sufficiently symmetric wave function? What are the charges of each particle (there's one particle for each j_z)? These are called the Δ particles.

(f) It's a bit more annoying to construct spin-$\frac{1}{2}$ particles, but they exist:

$$|\text{isospin and spin}\rangle = \frac{1}{\sqrt{2}}\left|\frac{1}{2}, j_z\right\rangle_{\text{A,isospin}} \left|\frac{1}{2}, s_z\right\rangle_{\text{A,spin}} + \frac{1}{\sqrt{2}}\left|\frac{1}{2}, j_z\right\rangle_{\text{B,isospin}} \left|\frac{1}{2}, s_z\right\rangle_{\text{B,spin}}. \tag{13}$$

Predict a charge e and charge 0 particle, each of spin-$\frac{1}{2}$. These are the proton and the neutron!

[3] *Hint:* Argue by "symmetry" that you only need to do this once, say at $S_z = \frac{1}{2}$. There are three vectors in Hilbert space, $|+\rangle|+\rangle|-\rangle$, $|+\rangle|-\rangle|+\rangle$ and $|-\rangle|+\rangle|+\rangle$. You've found two vectors in this subspace already. What's left?