Exam 1

Due: 11:59 PM, Thursday, October 1. Submit your exam via Canvas.

- You are allowed to use any course materials (including posted solutions), any books, and online references such as Wikipedia for help on this exam. **You must cite** every reference that you use (except course materials and assigned books) in an honest manner; failing to do so is considered academic dishonesty. **Do not collaborate** with any human, or ask for help via PhysicsForums, Chegg, Quora or any similar website. This is a very severe violation of the Honor Code. You may ask the instructor alone for help in the form of clarifying questions.
- The exam is intended to take no more than 4 hours, though there is no limit on the amount of time you can spend on it, within the allowed time period. Good luck!

Problem 1 (Coherent states): Consider a one dimensional particle of mass m in a harmonic oscillator of frequency ω. Let p_0, x_0 be real constants. Consider a particle in the initial state

$$\psi(x,0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}(x-x_0)^2 + \frac{\mathrm{i}p_0 x}{\hbar}\right]. \tag{1}$$

5 points (a) Apply the lowering operator a to the state $\psi(x,0)$. Show that

$$a\psi(x,0) = \left[\sqrt{\frac{m\omega}{2\hbar}}x_0 + \frac{\mathrm{i}p_0}{\sqrt{2m\omega\hbar}}\right]\psi(x,0). \tag{2}$$

5 points (b) Letting $|n\rangle$ denote the n^{th} excited oscillator state, consider the state

$$|\alpha\rangle = \sum_{n=0}^{\infty} \mathrm{e}^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{3}$$

Here α is a complex number. This is called a **coherent state**. Show that

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{4}$$

Hence, coherent states are eigenvectors of a of eigenvalue α.

5 points (c) Using (3), solve the time-independent Schrödinger equation to show that the time evolution of a coherent state is given by

$$|\alpha(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht/\hbar}|\alpha\rangle = \mathrm{e}^{-\mathrm{i}\omega t/2}|\alpha\mathrm{e}^{-\mathrm{i}\omega t}\rangle. \tag{5}$$

Hence, coherent states remain coherent for all times. (The overall phase factor cannot be physically measured, and can for practical purposes be ignored.)

(d) What is the position space wave function $\psi(x,t)$, given the initial condition (1)? You do not need to explicitly check that the position-space Schrödinger equation is satisfied, but do justify your answer.

10 points **Problem 2:** Consider a particle in two dimensions with Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2\left(x^2 + 4y^2\right). \tag{6}$$

Show that the allowed energy levels of this Hamiltonian are

$$E_n = \hbar\omega\left(n + \frac{3}{2}\right), \quad (n = 0, 1, 2, \ldots). \tag{7}$$

What is the degeneracy of energy level E_n?

10 points **Problem 3 (Alkenes):** Figure 1 shows a number of different molecules called **alkenes**, built out of carbon and hydrogen atoms. We only show the bonds between the carbon (C) atoms in the figure. While in chemistry one often depicts such molecules with alternating double and single carbon-carbon bonds, a better model for each molecule is that one electron (per carbon atom) freely floats around the entire molecule. We can crudely model a molecule with n carbon atoms as an n-level system (with n occupiable "hybrid orbitals" shared among carbon atoms). For linear chain (end Cs have one neighboring C) or ring (all Cs have two neighboring Cs) shaped molecules, one approximates the allowed energy levels

$$E_k^{\text{line}} = \eta\cos\frac{\pi k}{n+1}, \quad (k = 1, 2, \ldots, n,\ \text{line-like molecule with } n \text{ C atoms}), \tag{8a}$$

$$E_k^{\text{ring}} = \eta\cos\frac{2\pi k}{n}, \quad (k = 1, 2, \ldots, n,\ \text{ring-like molecule with } n \text{ C atoms}). \tag{8b}$$

where η is a "universal" constant that is common among all the alkenes.

(a) Using the spin and indistiguishability of the electrons, and assuming they are non-interacting, calculate the ground state energy of all 5 molecules shown in Figure 1. You should find approximately from highest to lowest: $-\eta$, -2η, -2.24η, -3.5η, -4η.

(b) Compare the ground state energies of one butadiene (C_4H_8) molecule to two ethylene (C_2H_4) molecules. Is it likely for butadiene to be unstable to breaking into two ethylene molecules?

(c) Repeat part (b) for cyclobutadiene. Does it make sense why cyclobutadiene is less chemically stable than butadiene?

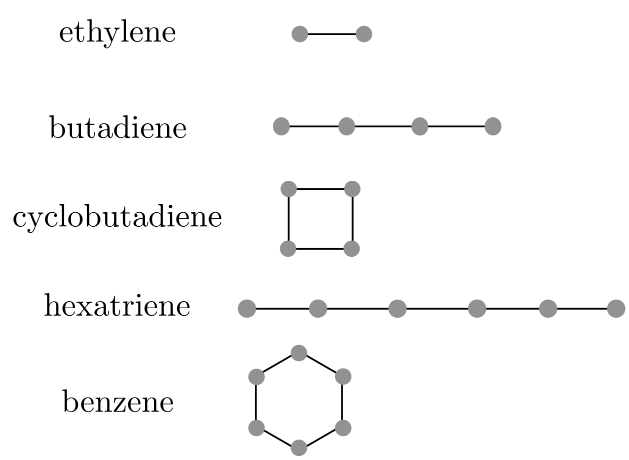

Figure 1: The alkene molecules. C atoms are gray.

(d) Compare the ground state energies of hexatriene to benzene. Which molecule would you expect to be more stable?

10 points **Problem 4:** How many different ways are there to place 3 non-interacting spin-0 bosons of mass m in a one dimensional infinite square well of length L, assuming the total energy

$$E \leq \frac{5\pi^2\hbar^2}{mL^2}? \tag{9}$$

Write down normalized wave functions for each possibility. You should find there are 3 linearly independent possibilities.

Problem 5: Consider a pair of indistinguishable particles in a two-level system (single particle basis states are $|1\rangle$, $|2\rangle$). Define the single-particle operator

$$Q = |1\rangle\langle 1| - |2\rangle\langle 2|. \tag{10}$$

Let η be a real constant. Consider the Hamiltonian

$$H = \eta Q \otimes Q, \tag{11}$$

where we define the product of operators as follows: for any single particle states $|\psi_1\rangle$ and $|\psi_2\rangle$:

$$(Q \otimes Q)(|\psi_1\rangle \otimes |\psi_2\rangle) = Q|\psi_1\rangle \otimes Q|\psi_2\rangle. \tag{12}$$

Note that we might also write $H = \eta Q_1 Q_2$ (this notation is in McIntyre), with the subscripts denoting which particle is acted on.

5 points (a) Show that the eigenvectors of H are $|1\rangle \otimes |1\rangle$, $|1\rangle \otimes |2\rangle$, $|2\rangle \otimes |1\rangle$ and $|2\rangle \otimes |2\rangle$. Show that two of these eigenvectors have energy $-\eta$ and two have energy η.

5 points (b) Show that you can find eigenvectors of H that are symmetric and/or antisymmetric under particle exchange. Here the particle exchange operator P acts as follows: $P|\psi_1\rangle \otimes |\psi_2\rangle = |\psi_2\rangle \otimes |\psi_1\rangle$.[1]

(c) Write down all degenerate ground states of this model if each indistinguishable particle is a spin-$\frac{1}{2}$ fermion. Note that you will now have to include spin in the wave function – $|1\rangle$ and $|2\rangle$ do not denote spin states; also note that H, given in (11), does not depend on spin.

[1] *Hint:* In this case, it might be easiest to simply look at the eigenvectors from part (a) and look for ways to make symmetric/antisymmetric combinations of eigenvectors (with the same eigenvalue!).