Exam 2

Due: 11:59 PM MST, November 10. Submit your exam via Canvas.

- You are allowed to use any course materials (including posted solutions), any books, and online references such as Wikipedia for help on this exam. **You must cite** every reference that you use (except course materials and assigned books) in an honest manner; failing to do so is considered academic dishonesty. **Do not collaborate** with any human, or ask for help via PhysicsForums, Chegg, Quora or any similar website. This is a very severe violation of the Honor Code. You may ask the instructor alone for help in the form of clarifying questions. You may use `Mathematica` or similar programs to help you do integrals and any symbolic manipulation.

Problem 1: Let $|n\rangle$ denote states parameterized by integers $n = 0, \pm 1, \pm 2, \ldots$. Consider the Hamiltonian defined by

$$H|n\rangle = \alpha|n+2\rangle + \alpha|n-2\rangle + \beta|n+3\rangle + \beta|n-3\rangle. \tag{1}$$

5 points (a) Let $T|n\rangle = |n+1\rangle$. Show that $[H, T] = 0$.

5 points (b) Find the eigenvectors and eigenvalues of H.

Problem 2: Consider a particle of mass m in the potential $V(x) = F|x|$, with $F > 0$. In this problem, we will use the variational principle to bound the ground state energy. Use the trial wave function

$$\psi_{\text{trial}}(x) = A \times \begin{cases} \ell - |x| & |x| \le \ell \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

with $A > 0$ and $\ell > 0$ constants.

5 points (a) By requiring that the wave function is normalized, show that

$$A = \sqrt{\frac{3}{2\ell^3}} \tag{3}$$

5 points (b) Find the value of ℓ which minimizes $\langle \psi_{\text{trial}}|H|\psi_{\text{trial}}\rangle$, the expectation value of the Hamiltonian in the state (2). Then conclude that the ground state energy of this Hamiltonian is

$$E_0 \le \left(\frac{81\hbar^2 F^2}{128m}\right)^{1/3}. \tag{4}$$

10 points **Problem 3:** Consider a 3 state system with Hamiltonian

$$H = \begin{pmatrix} 1+a & a & 0 \\ a & 2 & a \\ 0 & a & 3+a \end{pmatrix} \tag{5}$$

Suppose a is a perturbatively small parameter. Use time-independent perturbation theory to calculate the eigenvalues of H to order a^2.

10 points **Problem 4:** Consider a particle of spin $j = 1$, with angular momentum matrices J_x, J_y and J_z. Now consider following operator:

$$K = \cos\theta J_x + \sin\theta J_y. \tag{6}$$

(a) Explain how to write down J_x and J_y in the eigenbasis of J_z (the standard basis), by using the raising and lowering operators for angular momentum ($J_\pm$). Then use explicit formulas for $J_\pm$ to write down K as a 3×3 matrix.[1]

(b) What are the eigenvalues of K, and why?[2]

Problem 5: Consider 4 distinguishable spin-$\frac{1}{2}$ degrees of freedom, with the spin angular momentum operators $\mathbf{S}_{1,2,3,4}$ acting on each distinct spin. In the "uncoupled basis" the states are of the form $|m_1 m_2 m_3 m_4\rangle$ where $m_{1,2,3,4} = \pm\frac{1}{2}$ denotes the eigenvalue of $\hbar^{-1} S_{1z}$, etc.

5 points (a) Show that $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 0 \oplus 1 \oplus 1 \oplus 1 \oplus 2$. What does this expression mean?

5 points (b) Explain why there is a unique state in the Hilbert space $|\psi\rangle$, obeying

$$(\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4)^2 |\psi\rangle = 2(2+1)\hbar^2 |\psi\rangle, \tag{7a}$$

$$(S_{1z} + S_{2z} + S_{3z} + S_{4z})|\psi\rangle = \hbar|\psi\rangle. \tag{7b}$$

Find an expression for $|\psi\rangle$ in terms of the "uncoupled basis" of states $|m_1 m_2 m_3 m_4\rangle$.

5 points (c) Suppose that the particles have Hamiltonian

$$H = A\mathbf{S}_1 \cdot \mathbf{S}_2 + A\mathbf{S}_3 \cdot \mathbf{S}_4 \tag{8}$$

Use the fact that H is separable among two different sets of degrees of freedom to find the eigenvalues of H along with their degeneracies.

(d) Now suppose that H is given by

$$H = A\mathbf{S}_1 \cdot \mathbf{S}_2 + A\mathbf{S}_3 \cdot \mathbf{S}_4 + B(\mathbf{S}_1 + \mathbf{S}_2) \cdot (\mathbf{S}_3 + \mathbf{S}_4) \tag{9}$$

What are the eigenvalues of H and what are their degeneracies? You may assume that A and B are each "generic" constants, chosen so that there are no accidental degeneracies.

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[1] *Hint:* We've done this calculation before. You can follow what we did in the past when answering this question, but must show all of the intermediate steps of the calculation.

[2] *Hint:* If you think about this problem from the right perspective, no calculation is needed!