Exam 3

Due: 11:59 PM, Friday, December 11. Submit your exam via Canvas.

- You are allowed to use any course materials (including posted solutions), any books, and online references such as Wikipedia for help on this exam. **You must cite** every reference that you use (except course materials and assigned books) in an honest manner; failing to do so is considered academic dishonesty. **Do not collaborate** with any human, or ask for help via PhysicsForums, Chegg, Quora or any similar website. This is a very severe violation of the Honor Code. You may ask the instructor alone for help in the form of clarifying questions.

Problem 1: Consider two interacting particles of mass M moving on the surface of a sphere of radius R, with Hamiltonian

$$H = \frac{\mathbf{L}_1^2 + \mathbf{L}_2^2}{2MR^2} + B(L_{1z} + L_{2z}) + A\mathbf{L}_1 \cdot \mathbf{L}_2. \tag{1}$$

The operators $\mathbf{L}_{1,2}$ represent the orbital angular momentum of the particles only. Do not worry about particle indistinguishability in this problem. M, R, A and B are real and non-negative constants.

5 points (a) Show that when $A = 0$, the eigenvalues of H are

$$E_{\ell_1 m_1 \ell_2 m_2} = \frac{\hbar^2}{2MR^2}\left(\ell_1^2 + \ell_1 + \ell_2^2 + \ell_2\right) + B\hbar(m_1 + m_2) \tag{2}$$

where $\ell_{1,2} = 0, 1, 2, \ldots$, $m_1 = 0, \pm 1, \ldots, \pm \ell_1$, and $m_2 = 0, \pm 1, \ldots, \pm \ell_2$.

5 points (b) What are the eigenvalues of H when $A \neq 0$?

(c) Consider the following 2 new Hamiltonians:

$$H_1 = H + \alpha L_{1z}, \tag{3a}$$

$$H_2 = H + \alpha(L_{1x} + L_{2x}). \tag{3b}$$

Here α is a real constant. Without doing an explicit calculation, explain whether it will be easier to find the eigenvalues of H_1 or H_2, and why.

5 points **Problem 2:** Consider a particle hopping on a one dimensional lattice, with sites labeled by $|n\rangle$ for all the integers n. Let $T_\ell|n\rangle = |n + \ell\rangle$. Suppose that $[H, T_2] = 0$, where H is the Hamiltonian governing the single particle's motion.

(a) What is the Brillouin zone for such a system? Explain why you will always have two bands of energy levels inside of this Brillouin zone.

(b) In Lecture 12, we studied the Hamiltonian

$$H|n\rangle = \alpha|n\rangle - \beta|n-1\rangle - \beta|n+1\rangle. \tag{4}$$

Show that for this Hamiltonian, $[H, T_2] = 0$.

(c) We showed before that the eigenvalues of the H given in (4) are $E(\theta) = \alpha - 2\beta\cos\theta$, for $|\theta| \le \pi$. So it seems as if this Hamiltonian actually has only one band of eigenvalues. Reconcile this result with your argument from part (a).

Problem 3 (Slowly-varying perturbations): Consider a spin-$\frac{1}{2}$ particle with time-dependent Hamiltonian

$$H(t) = \frac{2}{\hbar}\left[BS^z + A\mathrm{e}^{-\gamma|t|}S^x\right], \tag{5}$$

Let $|+\rangle$ and $|-\rangle$ denote the up/down spin states in the z direction. Suppose that $|\psi(t\to-\infty)\rangle = |-\rangle$, and that $0 < A \ll B$.

5 points (a) Use first-order time-dependent perturbation theory to show that for times $t > 0$,

$$\langle +|\psi(t)\rangle \approx -\mathrm{i}A\left(\frac{1}{\hbar\gamma + 2\mathrm{i}B} + \frac{1-\mathrm{e}^{-(\gamma-2\mathrm{i}B/\hbar)t}}{\hbar\gamma - 2\mathrm{i}B}\right). \tag{6}$$

5 points (b) Suppose that we ignored the time dependence in the Hamiltonian $H(t)$. Use first-order time-independent perturbation theory to obtain the ground state of $H(t)$ at each fixed value of t.

(c) Argue that when $\gamma \ll \gamma_*$, $|\psi(t)\rangle$ is approximately in the ground state of $H(t)$ at every time t. Explain why this happens. Give an explicit formula for γ_*.

Problem 4: Consider the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + gx^4. \tag{7}$$

Assume that $m, \omega, g > 0$.

10 points (a) Use first order perturbation theory to find the eigenvalues of H to first order in g:[1]

$$E_n \approx \hbar\omega\left(n + \frac{1}{2}\right) + \frac{3g\hbar^2}{4m^2\omega^2}\left(2n^2 + 2n + 1\right). \tag{8}$$

(b) Explain why at large enough values of n, perturbation theory must break down.

5 points (c) Use the Bohr-Sommerfeld approximation to estimate the eigenvalues E_n for very large n to be[2]

$$E_n \approx g\left[\frac{\pi\hbar n}{C\sqrt{2mg}}\right]^{4/3}, \tag{9}$$

where $C \approx 1.75$ is a constant that arises from the integral

$$\int_{-1}^{1}\mathrm{d}x\sqrt{1-x^4} = C. \tag{10}$$

5 points (d) Let's define the function $E_0(g)$ to be the ground state energy of $H(g)$, the Hamiltonian defined in (7) with g thought of as a variable parameter. Show that $E_0(g)$ is a non-decreasing function of g: e.g. if $g_1 \le g_2$, then $E_0(g_1) \le E_0(g_2)$.

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[1] *Hint:* If A is a physical observable (Hermitian operator), then $\langle u|A^2|u\rangle = \langle v|v\rangle$, where $|v\rangle = A|u\rangle$. $|u\rangle$ or $|v\rangle$ need not be normalized quantum states – this is just a mathematical trick that will spare you some algebra.

[2] *Hint:* First, explain why you can approximate $\omega = 0$ for this part. When n is large, $n \pm \frac{1}{2} \approx n$.

Problem 5: Consider two interacting particles of mass m moving on a one dimensional ring of radius R, with Hamiltonian

$$H = -\frac{\hbar^2}{2mR^2}\left(\frac{\partial^2}{\partial\theta_1^2} + \frac{\partial^2}{\partial\theta_2^2}\right) + u\cos(\theta_1+\theta_2). \tag{11}$$

5 points (a) First, suppose $u = 0$. Check that the functions

$$\psi_{n_1 n_2}(\theta_1, \theta_2) = \frac{\mathrm{e}^{\mathrm{i}n_1\theta_1 + \mathrm{i}n_2\theta_2}}{2\pi} \tag{12}$$

are eigenfunctions of H. Here $n_{1,2}$ are integers. What are the corresponding eigenvalues? You do *not* need to show that these all eigenfunctions of H are of the form above (though this is indeed the case, and you may use this result in subsequent parts).

(b) If the particles are bosons with spin 0, show that there are 2 allowable (and orthogonal) states for the two particles to be in, with total energy

$$E = \frac{\hbar^2}{2mR^2}. \tag{13}$$

Write out the normalizable wave function $\psi(\theta_1, \theta_2)$ for each possibility.

5 points (c) Now consider the more general case, where $u \neq 0$. Use degenerate first order perturbation theory to show that the degenerate energy levels from part (b) become[3]

$$E_\pm \approx \frac{\hbar^2}{2mR^2} \pm \frac{u}{2}, \tag{14}$$

so long as u is sufficiently small.

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[3] *Hint:* If you do integrals by hand, use the identities $\cos\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}$ and $\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\mathrm{e}^{\mathrm{i}n\theta} = \delta_{n,0}$ when n is an integer.