Practice Exam 1

- You are allowed to use any of your class notes, books, and any document posted to Canvas. You are allowed to use any books and online references such as Wikipedia for help on this exam. **You must cite** every external reference that you use in an honest manner; failing to do so is considered academic dishonesty. **Do not collaborate** with any human, or ask for help via PhysicsForums, Chegg, Quora or any similar website. This is a very severe violation of the Honor Code. You may ask the instructor alone for help in the form of clarifying questions.
- The exam is intended to take no more than 4 hours, though there is no limit on the amount of time you can spend on it, within the allowed time period. Good luck!

Problem 1: In dimensionless units ($\hbar = m = \omega = 1$) the wave functions of the harmonic oscillator are

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}\pi^{1/4}} \mathrm{H}_n(x) \mathrm{e}^{-x^2/2}. \tag{1}$$

5 points (a) Apply the lowering operator to ψ_n to derive the identity

$$\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{H}_n(x) = 2n \mathrm{H}_{n-1}(x). \tag{2}$$

5 points (b) Apply the raising operator to ψ_n, and use (2), to derive the identity

$$\mathrm{H}_{n+1}(x) = 2x \mathrm{H}_n(x) - 2n \mathrm{H}_{n-1}(x). \tag{3}$$

5 points **Problem 2:** For the n^{th} energy level of the harmonic oscillator, calculate $\langle n | x^3 | n \rangle$.

Problem 3: Consider a pair of indistinguishable spin-1 particles. A single particle can exist in (superpositions of) states $|+\rangle$, $|0\rangle$ or $|-\rangle$.

5 points (a) Relating particle spin to its statistics, determine whether these particles are bosons or fermions.

(b) Which of the following wave functions are allowed, given the result of part (a)? Why?

$$|a\rangle = |00\rangle, \tag{4a}$$

$$|b\rangle = |0+\rangle, \tag{4b}$$

$$|c\rangle = \frac{|+-\rangle + |-+\rangle}{\sqrt{2}}, \tag{4c}$$

$$|d\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}, \tag{4d}$$

$$|e\rangle = \frac{|+-\rangle + \mathrm{i}|-+\rangle}{\sqrt{2}}. \tag{4e}$$

5 points (c) Which of the following are acceptable Hamiltonians describing the interactions between the two particles? Why? Assume α, β are real constants.

$$H_1 = \alpha \left(S_1^z + S_2^z\right), \tag{5a}$$
$$H_2 = \beta \left(S_1^x S_2^x + S_1^y S_2^y\right), \tag{5b}$$
$$H_3 = \beta S_1^x S_2^y, \tag{5c}$$
$$H_4 = \beta S_1^x S_2^x + \alpha \left(S_1^y + S_2^y\right), \tag{5d}$$
$$H_5 = \alpha S_1^z + \beta \left(S_2^x\right)^2. \tag{5e}$$

Problem 4: Consider two indistinguishable particles on a ring (parameterized by angular coordinate θ), in the state

$$\psi(\theta_1, \theta_2) = \frac{\mathrm{e}^{\mathrm{i}n\theta_1 + \mathrm{i}m\theta_2} + \sigma \mathrm{e}^{\mathrm{i}m\theta_1 + \mathrm{i}n\theta_2}}{2\pi\sqrt{2}}. \tag{6}$$

where $\sigma = \pm 1$ are both allowed. Consider both possible values of σ in this problem. Assume $m \neq n$ are integers.

5 points (a) Show that in this state ψ,

$$\langle \cos(\theta_1 - \theta_2) \rangle = \begin{cases} \dfrac{\sigma}{2} & m = n \pm 1 \\ 0 & \text{otherwise} \end{cases}. \tag{7}$$

5 points (b) Suppose that the particles are spin-$\frac{1}{2}$ particles in an angular state with one particle having $n = 3$ and the other particle having $m = 4$. Looking for total wave functions of the form $|\psi(\theta_1, \theta_2)\rangle \otimes |\psi_{\mathrm{spin}}\rangle$, write down a valid wave function for the two particles where $\psi(\theta_1, \theta_2)$ is given by either of the wave functions in (6) – i.e. with either $\sigma = +1$ or $\sigma = -1$.

(c) If we turned on a small repulsive interaction between the two fermionic particles, which state would have lower energy? No explicit calculation is necessary (or appropriate).

Problem 5 (Nuclear magic numbers): It is an empirical fact that, at least for light nuclei consisting of A nucleons (either protons or neutrons), nuclei are particularly stable if $A = 2, 8, 20, 28, 50$, etc. For simplicity, suppose that all the nucleons were neutrons, which are spin-1/2 fermions. A simple model to try and understand this effect would be to approximate the nucleus as a quantum mechanical system with Hamiltonian

$$H = \sum_{j=1}^{A} \left[\frac{p_{x,j}^2 + p_{y,j}^2 + p_{z,j}^2}{2m} + \frac{1}{2} m\omega^2 \left(x_j^2 + y_j^2 + z_j^2\right) \right].$$

Note that $p_{x,j} = -\mathrm{i}\hbar \partial / \partial x_j$, etc. (x_j, y_j, z_j) represent the coordinates of neutron j.

10 points (a) First, determine the eigenstates and eigenvalues of a single spinless particle ($A = 1$) in the three-dimensional harmonic oscillator in (5). Show that the ground state is unique, while the n^{th} excited state is degenerate, with degeneracy

$$D(n) = \frac{(n+1)(n+2)}{2}. \tag{8}$$

5 points (b) Describe the ground state of (5) when $A \neq 1$. Now include the spin of the particles. You do not need to write down the explicit wave function, just write a sentence or two explaining the key properties.

(c) One might postulate that nuclei with a unique (non-degenerate) ground state are particularly stable. Does our simple model (along with this assumption) explain the sequence of nuclear magic numbers?