Practice Exam 2

- You are allowed to use any of your class notes, books, and any document posted to Canvas. You are allowed to use any books and online references such as Wikipedia for help on this exam. **You must cite** every external reference that you use in an honest manner; failing to do so is considered academic dishonesty. **Do not collaborate** with any human, or ask for help via PhysicsForums, Chegg, Quora or any similar website. This is a very severe violation of the Honor Code. You may ask the instructor alone for help in the form of clarifying questions.

10 points **Problem 1:** Consider a single particle hopping on a one dimensional lattice, with sites labeled by $|n\rangle$, for integer n. Consider the Hamiltonian

$$H|3n\rangle = \alpha\left(|3n+3\rangle + |3n-3\rangle\right), \tag{1a}$$

$$H|3n+1\rangle = \beta\left(|3n+4\rangle + |3n-2\rangle\right), \tag{1b}$$

$$H|3n+2\rangle = \gamma\left(|3n+5\rangle + |3n-1\rangle\right). \tag{1c}$$

Assume $\alpha \neq \beta \neq \gamma$ are all real parameters. Find the eigenvalues and eigenvectors of H.[1]

10 points **Problem 2:** Consider a particle on a ring ($0 \le \theta < 2\pi$), with Hamiltonian

$$H = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\theta^2} - u\cos\theta. \tag{2}$$

Assume that $I, u > 0$.

(a) Use the variational principle to find an upper bound on the ground state energy of this system, using the trial wave function

$$\psi_{\text{trial}}(\theta;\alpha) = \frac{1}{\sqrt{2\pi}}\left[\cos\alpha + \sin\alpha\cos\theta\right]. \tag{3}$$

Here α is a variational parameter, that you should assume is real.[2]

(b) Your answer from part (a) will not be very close to the true ground state energy at large values of u. At large u, the potential energy $-u\cos\theta$ should dominate the Hamiltonian. What do you think the ground state energy will be? Give as accurate of an answer as you can.

[1] *Hint:* Begin by drawing a picture of where a particle can hop to from a given lattice site. Your picture should suggest how to solve this problem. You will spend a lot of unnecessary time solving this problem using the general algorithm discussed in Lecture 14.

[2] The manipulations here should end up looking rather similar to a problem from Homework 8.

10 points **Problem 3:** Consider a spin 1 particle, with Hamiltonian

$$H = AJ_z + BJ_x^2. \tag{4}$$

Here J_x and J_z denote the standard angular momentum operators.

(a) Suppose that B is much smaller than A in magnitude. Calculate the eigenvalues of H to quadratic order in the small parameter B, using perturbation theory.

(b) Find the exact eigenvalues of H, and compare to your perturbative result.

Problem 4: Consider two particles with spin $j_1 = j_2 = 2$.

5 points (a) Find the eigenvalues (along with their degeneracies) of the Hamiltonian

$$H = A\mathbf{J}_1 \cdot \mathbf{J}_2. \tag{5}$$

5 points (b) Find the state $|j = 4, m = 3\rangle$ in terms of the uncoupled basis vectors $|j_1 m_1 j_2 m_2\rangle$. Do not simply quote the answer from a table of Clebsch-Gordan coefficients (but you can check your answer this way) – you must derive the result analytically!

5 points (c) Now suppose that these particles are, in fact, indistinguishable. Are they bosons or fermions?

(d) Define the particle exchange operator P by $P|2m_1 2m_2\rangle = |2m_2 2m_1\rangle$. Explain why $[P, \mathbf{J}_1 + \mathbf{J}_2] = 0$. You do not need to check this identity explicitly for all 3 components of the total angular momentum if you don't want, but at least show this identity holds for the z-component.

5 points (e) Show that $|j = 4, m = 3\rangle$ is an eigenvector of P with eigenvalue 1. Conclude that $P|j = 4, m\rangle = |j = 4, m\rangle$.

(f) Show that the coupled basis vectors $|j = 3, m\rangle$ are also eigenvectors of P – what is the eigenvalue?

(g) Assuming that the full wave function is simply $|jm\rangle$ (i.e. ignore any possible position dependence, etc., in the wave function), determine which of the states found in parts (e) and (f) are allowed wave functions, given your answer to part (c).