Practice Exam 3

- You are allowed to use any of your class notes, books, and any document posted to Canvas. You are allowed to use any books and online references such as Wikipedia for help on this exam. **You must cite** every external reference that you use in an honest manner; failing to do so is considered academic dishonesty. **Do not collaborate** with any human, or ask for help via PhysicsForums, Chegg, Quora or any similar website. This is a very severe violation of the Honor Code. You may ask the instructor alone for help in the form of clarifying questions.

Problem 1: Consider two indistinguishable spin-1 bosons.

5 points (a) Describe in one or two sentences the notion of the uncoupled and coupled bases in the problem of angular momentum addition. Then, explain the formula $1 \otimes 1 = 0 \oplus 1 \oplus 2$.

(b) After a calculation (that you do not need to do), you could show that in the coupled basis, the two-particle spin wave functions with total spin 0 and 2 are symmetric, while the two-particle spin wave functions with total spin 1 are antisymmetric. By counting the total number of orthogonal states of two coupled spin-1 particles that are either symmetric or antisymmetric, show that this decomposition is plausible: e.g., that the number of antisymmetric states matches the Hilbert space dimension for a spin 1 system.

5 points (c) Now, suppose that these bosons are placed in a one-dimensional harmonic oscillator, with Hamiltonian

$$H_0 = \frac{p_1^2 + p_2^2}{2m} + \frac{m\omega^2}{2}\left(x_1^2 + x_2^2\right). \tag{1}$$

What are the eigenvectors of H_0, and what are their eigenvalues? You may express your answer in terms of single-particle eigenstates $|n\rangle$ of the harmonic oscillator, and/or single particle eigenfunctions $\psi_n(x)$. However, you must write down exact formulas for the energy levels of all states. Be careful to account for spin, along with any symmetry requirements on the total wave function due to the bosonic degrees of freedom!

5 points (d) Now, consider applying the perturbation

$$H' = \epsilon \mathbf{S}_1 \cdot \mathbf{S}_2 \tag{2}$$

where $\mathbf{S}_{1,2}$ represent the two spin operators for the 2 spin-1 particles. Find the eigenvalues of $H = H_0 + H'$ (you can do this exactly). How much does this perturbation lift any degeneracies that you found in part (c)?

10 points **Problem 2:** Consider two distinguishable spin-$\frac{1}{2}$ degrees of freedom, interacting via the Hamiltonian

$$H = \frac{4C}{\hbar^2} S_{1z}S_{2z} + \frac{4\epsilon}{\hbar^2} A S_{1x}S_{2x} + \frac{2\epsilon}{\hbar} B S_{1x}. \tag{3}$$

Using degenerate perturbation theory, find the eigenvalues of H to second order in ϵ.

5 points **Problem 3:** For each of the following Hamiltonians acting on particles on a one dimensional line, parameterized by $|n\rangle$for $n = \ldots, -1, 0, 1, \ldots$, find the smallest value of ℓ for which T_ℓ, defined by $T_\ell|n\rangle = |n+\ell\rangle$, commutes with the Hamiltonian: $[H, T_\ell] = 0$.

$$H_1|n\rangle = \begin{cases} \alpha|n\rangle & n = 3k, \\ \beta|n\rangle & n = 3k+1 \\ \gamma|n\rangle & n = 3k+2 \end{cases}, \tag{4a}$$

$$H_2|n\rangle = \begin{cases} \alpha|n-1\rangle + \beta|n+1\rangle & n = 3k, \\ \beta|n-1\rangle + \gamma|n+1\rangle & n = 3k+1 \\ \gamma|n-1\rangle + \alpha|n+1\rangle & n = 3k+2 \end{cases}, \tag{4b}$$

$$H_3|n\rangle = \alpha|n\rangle - \beta(|n+1\rangle + |n-1\rangle) - \gamma(|n+2\rangle + |n-2\rangle), \tag{4c}$$

$$H_4|n\rangle = \alpha \cos\frac{\pi n}{5}|n\rangle - \beta(|n+1\rangle + |n-1\rangle). \tag{4d}$$

Assume $\alpha \neq \beta \neq \gamma$. In the formulas above, k represents an integer. You do not need to do an explicit calculation if you can (briefly) explain your answer.

5 points **Problem 4:** Consider N bosons that move on a one dimensional ring of radius R, with Hamiltonian

$$H = -\frac{\hbar^2}{2mR^2}\sum_{n=1}^{N}\frac{\partial^2}{\partial\theta_n^2} + \lambda\sum_{m<n} V_2(\theta_m - \theta_n). \tag{5}$$

Here θ_n denotes the angular coordinate of the n^{th} particle. $V_2(\phi) = V_2(-\phi)$ is an even function that describes the potential energy between two interacting bosons; here ϕ is a dummy angular variable characterizing the difference between the angles of two particles, and the function $V_2(\phi)$ is the same function for every single pair of particles.

(a) First, suppose $\lambda = 0$. Explain why the ground state energy of the Hamiltonian is $E = 0$.

(b) Now, consider the case $\lambda > 0$, corresponding to interacting bosons. Explain why if the interactions are always attractive, $V_2(\phi) < 0$, then the ground state energy $E_0 < 0$.

Problem 5 (Spectroscopy of pigment molecules): On Homework 4, we saw that there are some mobile electrons in a pigment molecule, and that their motion is reasonably approximated by the motion of electrons in a one dimensional infinite square well of length L.

5 points (a) Write down the energy levels and corresponding stationary states for an electron in the infinite square well. Based only on this spectrum alone, what would you predict the frequencies of absorbed/emitted photons to be?

5 points (b) Now, assume that the dipole moment of the mobile electron is

$$\mathfrak{p} = -e\left(x - \frac{L}{2}\right), \tag{6}$$

where $0 \le x \le L$ is the particle's position in the well. Within the electric dipole approximation, which of the transitions you identified in part (a) become forbidden? In other words, what are the selection rules for this system? Explain why.[1]

[1] *Hint:* What happens to $\mathfrak{p}$, and to wave functions, if $x \to L - x$?

10 points **Problem 6 (Mesons):** On Homework 1, we looked at a toy model of mesons – elementary particles made up of a quark-antiquark pair, with a "constant tension force" pulling the pair together. As before, a toy model for this problem is to consider the one dimensional Hamiltonian

$$H = c|p| + F|x|, \tag{7}$$

where c is the speed of light and F is the force due to the flux tube. While quantizing this H is rather challenging using operators due to $|p|$ and $|x|$ both appearing in the Hamiltonian, we *can* apply the Bohr-Sommerfeld approximate quantization here.

(a) Sketch the curves of constant energy E in the (x, p) plane (called **phase space** in classical mechanics). Draw arrows along these curves to depict how the particle will move as a function of time (you do not need to find the time dependence, however).

(b) First state an algebraic formula for Bohr-Sommerfeld quantization, and then interpret it geometrically. (Although we only derived this condition for non-relativistic systems, you can assume it holds here as well.)

(c) Use Bohr-Sommerfeld quantization to approximately quantize this ultrarelativistic system: namely, predict the energy levels E_n.