PHYS 4410: Quantum Mechanics 2 **Spring 2023**

Exam 1

- This exam is open book and open notes. You are not allowed to use any electronic devices besides a calculator, nor allowed to communicate at all about the questions on the exam with any other students for 24 hours after the exam ends.
- You have 2 hours to complete this exam (unless Disability Services has authorized a request for extended time: 50% extra time is 3 hours, 100% extra time is 4 hours). Good luck!

Problem 1: Consider a quantum harmonic oscillator (you can set $m = \hbar = \omega = 1$ if you want to) in the initial state

$$|\psi(0)\rangle = \frac{|1\rangle - |3\rangle}{\sqrt{2}}, \tag{1}$$

where $|n\rangle$ denotes the n^{th} excited state of the oscillator.

10 **A:** What is the time-evolved wave function $|\psi(t)\rangle$?

10 **B:** Calculate the uncertainty in position as a function of time:

$$\Delta x(t) = \sqrt{\langle\psi(t)|x^2|\psi(t)\rangle - \langle\psi(t)|x|\psi(t)\rangle^2}. \tag{2}$$

Problem 2: Consider the two-dimensional harmonic oscillator

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_1^2 x_1^2 + \frac{1}{2}m\omega_2^2 x_2^2. \tag{3}$$

10 **A:** What are the eigenvalues of H?

15 **B:** Now, suppose that x_1 and x_2 should represent the positions of two indistinguishable spin-0 quantum mechanical particles, each placed in the same physical one-dimensional potential.

- B1. Why must we take $\omega_1 = \omega_2$ in order to interpret the particles as indistinguishable?
- B2. Write down an explicit wave function for the ground state in bra-ket notation, and in terms of a wave function that depends on x_1 and/or x_2. (You can use $\psi_n(x)$ to denote the energy states of the 1d oscillator.)
- B3. What is the smallest eigenvalue of H which is degenerate? Write down all degenerate eigenvectors for this energy.

15 **C:** Repeat part **B**, now assuming the particles are spin-$\frac{1}{2}$, and assuming that H does not depend on spin.

- C1. Write down an explicit wave function for the ground state.
- C2. What is the smallest eigenvalue of H which is degenerate? Write down all degenerate eigenvectors for this energy.

Problem 3: Consider two indistinguishable spin-0 particles, each of which can occupy 3 states: $|1\rangle$, $|2\rangle$, $|3\rangle$. For constant $\epsilon > 0$, define the single-particle Hamiltonian

$$H_0 = \epsilon|1\rangle\langle 1| + 2\epsilon|2\rangle\langle 2| + 3\epsilon|3\rangle\langle 3|. \tag{4}$$

10 **A:** Suppose that $H = H_0 \otimes 1 + 1 \otimes H_0$. Write down all eigenvalues and eigenvectors of H. Make sure to only include physical wave functions, given the spin of the particles.

10 **B:** Assuming $U > 0$, describe, as a function of U, the ground state of the following Hamiltonian:[1]

$$H = H_0 \otimes 1 + 1 \otimes H_0 + U\left[|1\rangle\langle 1| \otimes |1\rangle\langle 1| + |2\rangle\langle 2| \otimes |2\rangle\langle 2| + |3\rangle\langle 3| \otimes |3\rangle\langle 3|\right]. \tag{5}$$

Problem 4 (Covalent bonds): A toy model for the formation of a diatomic molecule X_2, with a covalent bond between two X atoms, is as follows (see Figure 1). Each X atom is modeled as an infinite square well of length L. If X has atomic number Z (i.e. there are Z protons in the nucleus), then we estimate the ground state energy of the atom as the ground state energy of the well with Z spin-$\frac{1}{2}$ electrons placed inside. Assume the electrons are non-interacting particles.

Similarly, model the diatomic molecule X_2 as an infinite square well of length $2L$, with $2Z$ electrons placed in their ground state in the well.

10 **A:** Suppose that $Z = 1$, i.e. X is H (hydrogen). Compare the ground state energies of two X atoms, vs. the ground state energy of a single X_2. Which is smaller? Do you expect that the H_2 molecule will be more stable than H?

10 **B:** Repeat for $Z = 2$, i.e. X is He (helium). Do you predict He_2 is more stable than He? Is the model very good?

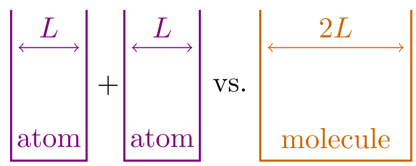

Figure 1: Toy model for the energies of atoms vs. molecules.

Problem 5: Consider a quantum mechanical particle in two dimensions described by the Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{k}{4}\left(x^4 + y^4\right). \tag{6}$$

5 **A:** There are "3 different parity symmetries" of H; namely, three (independent) transformations $P_{1,2,3}$, each of which obeys the necessary relations to be a "parity symmetry". What are $P_{1,2,3}$? For example, you may wish to state the wave function $P_1\psi(x,y)$ etc.

5 **B:** Another system with "3 different parity symmetries" is the following three dimensional model:

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + ax^2 + by^4 + cz^6. \tag{7}$$

Do you expect Hamiltonians (6) and (7) to have the same symmetry group? Why or why not?

10 **Problem 6:** Consider a spin-$\frac{1}{2}$ particle, trapped in a harmonic oscillator potential, with the following Hamiltonian (we have set $\hbar = m = \omega = 1$ in this problem):

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \lambda x \sigma^x \tag{8}$$

The last term explicitly couples the spin of the particle to its position: here $\sigma^x = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|$ flips the spin of the electron from up to down and vice versa.

Find the eigenvalues of H.

[1] *Hint:* Do the eigenstates of H differ between parts **A** and **B**?