PHYS 4410: Quantum Mechanics 2 **Spring 2023**

Exam 2

- This exam is open book and open notes. You are not allowed to use any electronic devices besides a calculator, nor allowed to communicate at all about the questions on the exam with any other students for 24 hours after the exam ends.
- You have 2 hours to complete this exam (unless Disability Services has authorized a request for extended time: 50% extra time is 3 hours, 100% extra time is 4 hours). Good luck!

Problem 1: Consider a particle of spin $j_1 = \frac{3}{2}$, and a particle of spin $j_2 = 1$. The angular momentum operators for each of these will be denoted as $\mathbf{J}_1$ and $\mathbf{J}_2$.

10 **A:** What is $\frac{3}{2} \otimes 1$ equal to? Explain what the physical importance of this is, in a few sentences.

15 **B:** Show that in the coupled basis, the state

$$\left|\frac{3}{2}\frac{3}{2}\right\rangle = \sqrt{\frac{3}{5}}\left|\frac{3}{2}\frac{3}{2}10\right\rangle - \sqrt{\frac{2}{5}}\left|\frac{3}{2}\frac{1}{2}11\right\rangle. \tag{1}$$

You cannot consult a table of Clebsch-Gordan coefficients; explain how this result can be derived!

15 **C:** Let $A > 0$ be a constant. Suppose that these two particles interacted with Hamiltonian

$$H = A\mathbf{J}_1 \cdot \mathbf{J}_2. \tag{2}$$

C1. Explain why the state in (1) would be an eigenstate of H. What is its eigenvalue E?

C2. What is the degeneracy of energy E, for Hamiltonian H?

Problem 2: Consider a particle in the infinite square well, with Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \qquad V(x) = \begin{cases} 0 & 0 \le x \le L \\ \infty & \text{otherwise} \end{cases}. \tag{3}$$

20 **A:** Consider the trial wave function

$$\psi(x) = \begin{cases} Ax(L-x) & 0 \le x \le L \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

A1. Find the normalization constant A.

A2. Use the variational principle with this as a trial wave function: namely, evaluate $\langle\psi|H|\psi\rangle$ and obtain a bound on the ground state energy.

5 **B:** What is the normalized trial wave function $\psi(x)$ for which $\langle\psi|H|\psi\rangle$ would take the lowest possible value? Give an explicit formula for your answer, and explain.

20 **Problem 3:** Let $a, b > 0$ be constants. Consider the following Hamiltonian describing a particle moving in a one-dimensional lattice:

$$H = -\sum_{n=-\infty}^{\infty} \left(a \left[|n+2\rangle\langle n| + |n\rangle\langle n+2| \right] + b \left[|n+3\rangle\langle n| + |n\rangle\langle n+3| \right] \right). \tag{5}$$

1. If R is the discrete translation operator (Lecture 11), show that $[H, R] = 0$.

2. Deduce the eigenvalues and eigenvectors of H.

Problem 4: Consider a particle moving on a sphere, with orbital angular momentum vector $\mathbf{L}$, and Hamiltonian (here $A > 0$ is a real constant)

$$H = A\mathbf{L}^2 + 3\hbar A L_x. \tag{6}$$

15 **A:** What are the eigenvalues of H?

10 **B:** Find the ground state wave function in the basis $|lm\rangle$, where l is the total orbital angular momentum, and $L_z|lm\rangle = \hbar m|lm\rangle$. This is the standard basis discussed in, e.g., Lecture 15.

10 **Problem 5:** Find the eigenvalues of the following Hamiltonian:

$$H = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{7}$$