PHYS 4410: Quantum Mechanics 2 **Spring 2023**

Exam 3

- This exam is open book and open notes. You are not allowed to use any electronic devices besides a calculator, nor allowed to communicate at all about the questions on the exam with any other students for 24 hours after the exam ends (or until grades are released).
- You have 2 hours to complete this exam (unless Disability Services has authorized a request for extended time: 50% extra time is 3 hours, 100% extra time is 4 hours). Good luck!

Problem 1: Consider a quantum mechanical particle of mass m in potential (take $a > 0$ and $V_0 > 0$)

$$V(x) = \begin{cases} 0 & |x| \le a \\ V_0 \left(1 - \dfrac{|x|-a}{a}\right) & |x| > a \end{cases}. \tag{1}$$

15 **A:** Since the potential $V(x \to \pm\infty) \to -\infty$, we know that there cannot be any true bound states. However, we might expect that bound states exist in this potential on a relatively long time scale τ.

- A1. Use the Bohr-Sommerfeld approximation to estimate the lowest "bound state" energy E of this potential. You may assume for now that V_0 is "large", so that this approximation is accurate.
- A2. Estimate the number of "bound states" in this potential.

15 **B:** Estimate the time τ over which a particle of energy $E < V_0$ can tunnel through V and escape to $x = \pm\infty$.

Problem 2: Consider a harmonic oscillator with a time-dependent frequency:

$$H(t) = \frac{p^2}{2m} + \frac{1}{2} m \mathit{\Omega}(t)^2 x^2, \tag{2}$$

where (assume in this problem that $2\omega_0 \ne \alpha$, and that δ is very small):

$$\mathit{\Omega}(t)^2 = \omega_0^2 + 2\delta \mathrm{e}^{-|t|/\tau}\cos(\alpha t) = \omega_0 + \delta\left(\mathrm{e}^{\mathrm{i}\alpha t} + \mathrm{e}^{-\mathrm{i}\alpha t}\right)\mathrm{e}^{-|t|/\tau}. \tag{3}$$

As $t \to -\infty$, we start in the ground state of the harmonic oscillator with frequency ω_0.

20 **A:** Use time-dependent perturbation theory, with H_0 the harmonic oscillator with frequency ω_0.

- A1. Show that there is only one eigenstate $|n\rangle$ ($n \ne 0$) of H_0 that the system will transition to, within first-order perturbation theory.
- A2. Show that

$$|\langle 0|\psi(t=\infty)\rangle|^2 \approx 1 - \frac{2\delta^2\tau^2}{\omega_0^2}\left[\frac{1}{1+(2\omega_0+\alpha)^2\tau^2} + \frac{1}{1+(2\omega_0-\alpha)^2\tau^2}\right]^2. \tag{4}$$

- A3. What would happen if $2\omega_0 = \alpha$? A qualitative explanation is fine.

10 **B:** Assume that $\alpha = 0$ for simplicity. Suppose that δ is no longer small, and so we cannot use perturbation theory. Under what conditions on τ can we nevertheless guarantee that with high probability, the oscillator is found in its ground state at $t = +\infty$? A "qualitative" answer is fine; don't worry about factors of 2, etc. here.

Problem 3: Consider a four-state quantum system with Hamiltonian

$$H = \begin{pmatrix} 0 & a & 0 & a \\ a & b + c\cos(\omega t) & a & 0 \\ 0 & a & 0 & a \\ a & 0 & a & -b - c\cos(\omega t) \end{pmatrix}. \tag{5}$$

10 **A:** Suppose that $b = c = 0$.

A1. Explain why (no calculation is needed, so long as your explanation demonstrates understanding) the eigenvectors of H are of the form

$$|\bar{k}\rangle = \frac{1}{2}\begin{pmatrix} \mathrm{e}^{-\pi \mathrm{i}k/2} \\ \mathrm{e}^{-\pi \mathrm{i}k} \\ \mathrm{e}^{-3\pi \mathrm{i}k/2} \\ 1 \end{pmatrix}, \quad (k = 1, 2, 3, 4). \tag{6}$$

A2. What are the eigenvalues of H associated with each eigenvector? You should find that two of these eigenstates are degenerate.

15 **B:** Now suppose that $c = 0$, but $b \neq 0$. Treat b as a perturbative parameter. What are the eigenvalues of H at first order in perturbation theory?[1]

15 **C:** If $c = 0$, show that the ground state energy is (at second order)

$$E \approx -2a - \frac{b^2}{4a}. \tag{7}$$

10 **D:** Now suppose that $b = 0$, but $c \neq 0$ is small. At what frequencies will the perturbation induce transitions between the energy levels of the unperturbed H with high probability?

10 **Problem 4:** Consider two indistinguishable spin-$\frac{1}{2}$ particles moving on a ring (the angular coordinates $\phi_{1,2}$ that describe them are periodic, i.e. ϕ_1 and $\phi_1 + 2\pi$ are the same point). In units where $\hbar = 1$, they are described by Hamiltonian

$$H = -4\epsilon\left(\frac{\partial^2}{\partial\phi_1^2} + \frac{\partial^2}{\partial\phi_2^2}\right) + 5\epsilon \mathbf{S}_1 \cdot \mathbf{S}_2. \tag{8}$$

1. What are the eigenvalues and eigenvectors of H?

2. Let the dipole moment operator be

$$\mathfrak{p} = a(S_{1x} - S_{2x})\sin(\phi_1 - \phi_2) \tag{9}$$

where a is a constant. If the interaction between this quantum system and photons is accurately described within the electric dipole approximation, what is the lowest energy photon that can be absorbed or emitted by this quantum system?

[1] *Hint:* What is the appropriate perturbation matrix V? Evaluate $\langle\bar{k}|V|\bar{k}'\rangle$ for all k, k' – you might notice a pattern after a few of these calculations.