Homework 1

Due: January 26 at 11:59 PM. Submit on Canvas.

Problem 1 (Superfluidity): As discussed in Lecture 1, the actual interaction between two atoms is a rather complicated function, which can be well approximated by a harmonic oscillator near its minimum. To ensure that algebra is tractable, let us suppose that the interaction potential between atoms is a rather peculiar function:

$$V(x) = U\left[\left(\frac{a}{x}\right)^8 - \left(\frac{a}{x}\right)^5\right]. \tag{1}$$

Here $a \approx 10^{-10}$ m is an atomic length scale, while $U \approx 2 \times 10^{-23}$ J is the energy scale of binding between the neutral atoms. In this problem, assume that the coordinate $x > 0$. The Hamiltonian of the system is

$$H = \frac{p^2}{2m} + V(x). \tag{2}$$

20 **A:** Let us first postulate that it is reasonable to approximate the minimum of the potential well with a harmonic oscillator.

- A1. Find the location, x_0, at which $V(x)$ has a local minimum. Show that $x_0 = c_1 a$, where $c_1 > 0$ is a dimensionless constant (such as $2, \sqrt{3}, \pi$, etc.).
- A2. What is $V(x_0)$? Show that $V(x_0) = -c_2 U$, where $c_2 > 0$ is a dimensionless constant.
- A3. Next, Taylor expand
$$V(x) \approx V(x_0) + \frac{k}{2}(x - x_0)^2, \tag{3}$$
and determine the constant $k = c_3 U/a^2$. (Again, c_3 is dimensionless.)
- A4. If (3) was exact, what would be the ground state energy of the quantum system?[1]

15 **B:** When your answer to A4 is that the ground state energy is close to (or above!) 0, the approximation (3) breaks down. In this limit, it's better to instead do a heuristic analysis following Lecture 1.

- B1. In the ground state, why might we estimate that
$$\Delta p^2 \sim \frac{\hbar^2}{4\Delta x^2}? \tag{4}$$
- B2. Following Lecture 1, look for the minimum of $H(\Delta x)$ using calculus, and the "exact" model for $V(x)$ given in (1). Show that there is a critical mass m_c, such that if $m < m_c$, there are no minima of $H(\Delta x)$ at finite Δx, and hence the atoms will not bind together.

[1] *Hint:* Plug in (3) into (2). You can quote the ground state energy of the harmonic oscillator, but you need to shift the answer by $V(x_0)$, and convert from the parameters in this problem to those from Lecture 2.

5 **C:** The lightest two long-lived bosonic nuclei, which are unlikely to form chemical bonds (either amongst themselves or with other elemental atoms), are helium-4 ($m \approx 4m_0$) and neon-20 ($m \approx 20m_0$), where $m_0 \approx 2 \times 10^{-27}$ kg is the mass of the proton. By numerically evaluating m_c given the estimates for U and a, argue that there will be no bound state for helium-4 atoms, while there would be for neon.

5 **D:** If you're still interested, go back and plot $H(\Delta x)$ as a function of increasing m. Do you think that m_c is actually the mass where the atoms prefer to be far separated, or is there a larger $m'_\mathrm{c} > m_\mathrm{c}$ where the atoms prefer to be unbound?

At ultra-low temperatures when quantum mechanical effects become important, an absence of any bound states suggests that the helium atoms will form a collective condensate where individual atoms are delocalized on macroscopic length scales. The resulting phase of matter is called a superfluid, and is indeed realized in helium-4 at low temperature! In contrast, neon becomes a solid upon cooling.

Problem 2 (Covalent bond): An extremely important system that can be modeled accurately by a simple harmonic oscillator is the chemical bond in a diatomic molecule A_2, with A = H, N, F, O, etc. If the A atom has mass m, then the Hamiltonian describing the simple harmonic oscillator is

$$H = \frac{p^2}{m} + \frac{1}{4}m\omega^2 x^2. \tag{5}$$

The funny constant factors above are deliberate, and arise because the harmonic oscillation describes the *relative motion* of the two atoms.

15 **A:** Let $|0\rangle$ denote the ground state of this oscillator. Evaluate the "size" D of this oscillator in the ground state, and show that

$$D = \sqrt{\langle 0|x^2|0\rangle} = \sqrt{\frac{\hbar}{m\omega}}. \tag{6}$$

A atom	m (10^{-27} kg)	ω (10^{12} Hz)	L (10^{-10} m)
H	1.7	827	0.7
N	23.4	438	1.1
Cl	58.5	104	2.0
Br	132	61	2.3

Figure 1: (Rough) experimental values of m, ω, and L for simple diatomic molecules.

10 **B:** The data in Figure 1 lists the experimentally determined bond length L of the covalent bond in the A_2 molecule. Is D or L larger? Does your answer make physical sense?

Problem 3 (Squeezed state): It is crucial that the ground state of the harmonic oscillator has $\Delta x > 0$ to have consistency with quantum mechanics. Still, it is possible to find states that have Δx below the ground state value. These states are called **squeezed**. They often play a valuable role in quantum sensing experiments and (future) technology.

20 **A:** An example of a state which can be squeezed is

$$|\psi\rangle = \alpha|0\rangle + \beta|2\rangle. \tag{7}$$

A1. Use raising and lowering operators to evaluate $\Delta x = \sqrt{\langle\psi|x^2|\psi\rangle - \langle\psi|x|\psi\rangle^2}$, assuming that the state is normalized ($|\alpha|^2 + |\beta|^2 = 1$), but otherwise arbitrary.

A2. Find choices of α and β which minimize

$$\Delta x = \sqrt{\langle\psi|x^2|\psi\rangle - \langle\psi|x|\psi\rangle^2}. \tag{8}$$

Show that the minimum value is *below* what it would be in the ground state ($\alpha = 1, \beta = 0$). You can do this numerically, not analytically, as long as your solution shows clear understanding.

B: If we start an oscillator in the initial state $|\psi\rangle$ above, we can then ask about its time evolution.

B1. Determine $|\psi(t)\rangle$. Explain why it always takes the form of (7), but with time dependent parameters $\alpha(t)$ and $\beta(t)$.

B2. Describe what happens to Δx as a function of time. Is the squeezing of $|\psi(0)\rangle$ robust?

Problem 4 (Bogoliubov transformation): In the emergent descriptions of superfluids or superconductors, one often finds (many-particle generalizations of) Hamiltonians of the form

$$H = \epsilon a^\dagger a - \eta \left(a^\dagger a^\dagger + aa \right). \tag{9}$$

where $a^\dagger/a$ are creation/annihilation operators of a harmonic oscillator: $[a, a^\dagger] = 1$. Take $\epsilon > 0$ and $\eta > 0$ to be real. Interestingly, if η is not too large, you can exactly solve this problem.

1. The **Bogoliubov transformation** defines a new set of creation/annihilation operators $b^\dagger$ and b:

$$b = a \cosh \alpha - a^\dagger \sinh \alpha, \tag{10a}$$

$$b^\dagger = a^\dagger \cosh \alpha - a \sinh \alpha. \tag{10b}$$

Here sinh and cosh are the hyperbolic trigonometric functions (which you can read about online if you aren't familiar). Show that b and $b^\dagger$ obey the "correct" commutation relation:

$$[b, b^\dagger] = 1. \tag{11}$$

2. In one or two sentences, explain how to find the eigenvalues of

$$H_0 = J b^\dagger b, \tag{12}$$

where J is some constant.

3. Show that for a clever choice of α, if you write (9) in terms of b and $b^\dagger$, it looks similar to the form (12). Thus determine the spectrum of the original Hamiltonian exactly, and determine the maximal value of η for which a solution exists.

4. Let $|n\rangle$ denote the original oscillator eigenstates: i.e. $a^\dagger a|n\rangle = n|n\rangle$. Find the ground state of H, as given in (9), in terms of the $|n\rangle$ basis.