PHYS 4410: Quantum Mechanics 2 **Spring 2023**

Homework 11

Due: April 27 at 11:59 PM. Submit on Canvas.

30 **Problem 1:** Consider the harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{1}$$

In this problem, it may help to work in units where $\hbar = m = \omega = 1$.

1. Sketch the curve of constant $H = E$ in the classical phase space. What does it look like?
2. Use Bohr-Sommerfeld quantization to approximately quantize H. Compare to the exact answer.

Problem 2: Consider a particle of mass m in a deep potential well in one dimension:

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \le x \le L \\ V_0 & x > L \end{cases}. \tag{2}$$

20 **A:** Let us use the Bohr-Sommerfeld approximation to estimate the energy levels.

A1. What are the energy levels E_n that you predict? Assume for the moment that n is relatively small.

A2. Compare your answer to the infinite square well (what should happen if $V_0 \to \infty$) and comment on the result.

10 **B:** What happens if $n \gg 1$? Argue that there are a finite number of bound states, with higher energy eigenstates being unbound. Estimate the number of bound states of the potential $V(x)$.

Problem 3 (α decay): In this problem we will study a simple model for the decay of a heavy nucleus by emitting an α particle, also known as a ^{4}He nucleus. For convenience, let us simplify our model of a nucleus to consist of a particle in the following one-dimensional potential for $x \ge 0$:

$$V(x) = \begin{cases} 0 & 0 \le x < a \\ \dfrac{2Ze^2}{4\pi\epsilon_0 x} & x \ge a \end{cases}. \tag{3}$$

Here a is the size of the nucleus, while Z is the number of protons in the nucleus (after the α decay). Let E be the kinetic energy of the particle trapped in the nucleus.

30 **A:** Let us consider a semiclassical model for the lifetime of this metastable state, following Lecture 36.

A1. Explain why the α particle "trapped inside" $0 \le x \le a$ is in a metastable state.

A2. Estimate the probability of the particle tunneling through the potential barrier during each "collision" with the wall at $r = a$. You may want to use `Mathematica` to evaluate an integral.

A3. Deduce your estimate for the lifetime of the metastable state, approximating that $E \ll Ze^2/4\pi\epsilon_0 a$ to simplify your answer to:

$$\tau \approx a\sqrt{\frac{2m}{E}} \exp\left[\frac{Ze^2}{\epsilon_0 \hbar}\sqrt{\frac{m}{2E}} - \frac{2}{\hbar}\sqrt{\frac{mZe^2 a}{\pi\epsilon_0}}\right]. \tag{4}$$

10 **B:** The typical size of a nucleus is about $a \sim 10^{-15} Z^{1/3}$ m, while the mass of an α particle is about 10^{-26} kg. Suppose that the energy it is ejected with is $E \sim 1.5 \times 10^{-14} Z$ J.

B1. Estimate the lifetime of a uranium atom with $Z \sim 92$.

B2. Argue that, within our very simple model, there is a maximal value of Z at which a nucleus might be stable. Compare to $Z \sim 120$, which is the largest nucleus created to date. (Note that all such nuclei are extremely unstable.)

20 **Problem 4 (Disorder):** In this problem, we will study the eigenstates of a Hamiltonian describing a one-dimensional particle moving through a weak random potential:

$$H = \frac{p^2}{2m} + V(x), \tag{5}$$

where $V(x)$ is random (in a manner we will specify later), but has small amplitude.

We look for an eigenfunction of H with energy E: call it $\psi(x)$. Defining

$$k = \frac{\sqrt{2mE}}{\hbar}, \tag{6}$$

we make a WKB-like ansatz for the wave function:

$$\psi(x) = R(x)\sin\theta(x), \tag{7a}$$

$$\frac{\mathrm{d}\psi(x)}{\mathrm{d}x} = kR(x)\cos\theta(x). \tag{7b}$$

If $V(x) = 0$, a solution to these equations is $\theta(x) = kx$ and $R(x) = 1$.

1. Show that the consistency of our definitions for R and θ, together with the time-independent Schrödinger equation, require:

$$\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}x} = \frac{mV}{k\hbar^2}\sin(2\theta), \tag{8a}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = k - \frac{mV}{k\hbar^2}\left(1 - \cos(2\theta)\right). \tag{8b}$$

2. Argue that at first order in the small number V, we can write

$$\theta(x) = kx - \int_0^x \mathrm{d}y \frac{mV(y)}{k\hbar^2}\left(1 - \cos(2ky)\right). \tag{9}$$

3. If V is random, and equally likely to be positive or negative, then such a first order correction in V is not very interesting, and might be negligible. However, if we find terms proportional to V^2, such terms could be important, because this is always positive. Along these lines, plug in (9) into your equation for R, and argue that at second order in V:

$$\log R(x) \approx -\frac{m}{E\hbar^2} \int_0^x \mathrm{d}y V(y) \cos(2ky) \int_0^y \mathrm{d}z V(z) \left(1 - \cos(2kz)\right). \tag{10}$$

4. Let $\mathbb{E}[\cdots]$ denote averages over the random disorder potential $V(x)$. If the disorder has very short-range correlations, we can argue that

$$\mathbb{E}[V(x)V(y)] \approx D\delta(x-y), \tag{11}$$

where D is a constant related to the "strength" of the disorder (weak disorder means D is "small"). Conclude that upon averaging over disorder, you expect that a typical wave function will have an *exponentially changing amplitude*

$$R(x) \sim \mathrm{e}^{x/\xi} \tag{12}$$

at large, positive, x. Find an expression for the **localization length** ξ.

Notice that this wave function a priori is very badly normalized: $R \to \infty$ as $x \to \infty$. The physical resolution to this problem is that our calculation is effectively probing the left tail of a wave function localized around some point x_0 far to the right, and if we found the true solution to the Schrödinger equation, we would see: $R(x) \sim \mathrm{e}^{-|x-x_0|/\xi}$. The location of x_0 would depend on precise details of $V(x)$ and is beyond the simple approximations made above.

The physical conclusion is as follows: even a tiny amount of disorder will have drastic consequences on the behavior of the eigenfunctions of H, which go from being delocalized plane waves at $D = 0$, to exponentially localized for any $D > 0$. This phenomenon is called **Anderson localization**, after its discoverer.