PHYS 4410: Quantum Mechanics 2 **Spring 2023**

Homework 3

Due: February 9 at 11:59 PM. Submit on Canvas.

Problem 1: Suppose I have two spin-$\frac{1}{2}$ particles, with Hamiltonian

$$H = A_1 S_{z,1} + A_2 S_{z,2}. \tag{1}$$

Here $S_{z,1/2}$ denotes the 2×2 matrix acting on a spin-$\frac{1}{2}$ degree of freedom.

15 **A:** Let us begin by describing the Hilbert space of this quantum system.

- A1. Write down the 4 possible states that the quantum system can be in (neglect particle indistinguishability until part **C**).
- A2. Write down 4×4 matrices representing $S_{z,1}$ and $S_{z,2}$.
- A3. Write down $S_{z,1}$ and $S_{z,2}$ as the tensor product of two appropriate 2×2 matrices, and explain in one sentence what the tensor product accomplishes for us here.

10 **B:** Define the particle exchange operator P as we did in Lecture 7. Show that $PHP = H$, or $[P, H] = 0$, only if $A_1 = A_2$.

10 **C:** From now on, assume $A_1 = A_2$, so that the particles are indistinguishable. Recall that since the particles are spin-$\frac{1}{2}$, they must be fermions.

- C1. If $|\psi\rangle$ is a fermionic quantum state describing these two particles' (spins), what is $P|\psi\rangle$?
- C2. How many fermionic states can you find?
- C3. What is the energy of each of these states?

20 **Problem 2:** Consider 3 spin-0 indistinguishable and non-interacting particles, capable of occupying the four quantum states $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$.

1. Are these particles bosons or fermions? Why?
2. Write down a wave function where 2 particles are in $|3\rangle$, and one particle is in $|1\rangle$. Is your answer unique?
3. How many (bosonic) states are there in the Hilbert space? (You don't have to write them all explicitly.)

Problem 3: Consider the Hamiltonian H and eigenstates $|n\rangle$ (or $\psi_n(x)$ if you prefer) of the one-dimensional infinite square well, discussed in McIntyre Section 5.3 (and from the first semester). Define the operator Q, which acts on wave functions as

$$Q\varphi(x) = \varphi(L - x). \tag{2}$$

5 **A:** Show explicitly that $Q^2 = 1$.[1]

[1] *Hint:* It might be easiest to do this by acting with each side of the operator equality on a general wave function $\varphi(x)$.

10 **B:** Show explicitly that $QHQ = H$, or $[Q, H] = 0$.

10 **C:** Classify the eigenstates ψ_n into either the even or odd representation of the $\mathbb{Z}_2$ symmetry.

Problem 4 (Pigments): A typical pigment molecule has the structure sketched in Figure 1, and consists of a long chain of covalently bonded atoms which can be approximated as a one dimensional infinite square well. Suppose there are $N+1$ atoms in the long chain of carbon atoms: approximate that each of the N bonds contributes one "free" spin-1/2 electron, of mass m, which may move up and down the chain of bonds freely. If each bond has length a, when N is large, we may thus approximate these electrons as moving in an infinite square well of width $L = Na$.

10 **A:** Using the Pauli exclusion principle and the energy levels for the particle in a box (i.e. infinite square well), describe which energy levels in the box are filled and which are empty in the ground state. Ignore electron-electron interactions.

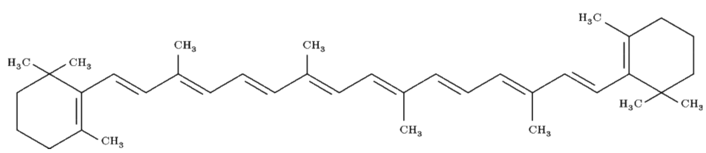

Figure 1: The β-carotene molecule is responsible for the orange color of carrots.

10 **B:** Now, suppose we send a photon of wavelength λ at the pigment molecule.

B1. What is the largest value of λ such that the photon can be absorbed by an electron in the pigment molecule? Assume $N > 1$. When the photon is absorbed, the electron must be able to jump to an unoccupied state in the box. You should find that when $N \gg 1$,

$$\lambda \approx \frac{4cma^2}{\pi\hbar} N. \tag{3}$$

B2. We might estimate that $N = 18$ for β-carotene, as depicted in Figure 1. Evaluate λ, given that $m \approx 9 \times 10^{-31}$ kg and $a \approx 10^{-10}$ m, and compare to the wavelength of orange light: 600 nm.

15 **Problem 5 (Cavity quantum electrodynamics):** In atomic physics, the interactions of an electron in an atom with photons in a cavity can be modeled as follows. We consider a single harmonic oscillator, where states $|n\rangle$ correspond to the quantum state with n photons excited, coupled to a single spin-$\frac{1}{2}$ degree of freedom (modeling two different quantum states of the atom) $|s\rangle$, where $s = \uparrow, \downarrow$. So the total Hilbert space consists of states of the form $|n\rangle \otimes |s\rangle$. Letting $a|n\rangle = \sqrt{n}|n\rangle$ denote the standard lowering operator, we consider the Hamiltonian

$$H = \omega_0 a^\dagger a + A\sigma_z + B\left(a\sigma_+ + a^\dagger\sigma_-\right), \tag{4}$$

where $\sigma_+ = |\uparrow\rangle\langle\downarrow|$, $\sigma_- = |\downarrow\rangle\langle\uparrow|$, $\sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$, and ω_0, A, B are constants. Note that here $a^\dagger a$ is shorthand for $a^\dagger a \otimes 1$ and σ_z is shorthand for $1 \otimes \sigma_z$: from the context it is clear what operators act on the oscillator vs. the spin.

1. Let $N = a^\dagger a + \frac{1}{2}\sigma_z$. Show that $[H, N] = 0$.

2. Use this fact to find exact formulas for all eigenvalues of H.