PHYS 4410: Quantum Mechanics 2 **Spring 2023**

Homework 6

Due: March 9 at 11:59 PM. Submit on Canvas.

Problem 1 (Nuclear magnetic moments): In nuclei that have an odd number of nucleons (protons or neutrons) inside of them, their magnetic moments are often very accurately predicted due to the following simple model. Suppose that the total z-magnetic moment, μ_z, is given by

$$\mu_z = AL_z + BS_z, \tag{1}$$

where A and B are constants. We assume that the entire magnetic moment of the nucleus comes from just the very last odd nucleon, which will generally not have a well-defined L_z or S_z, but will be in a state of fixed $J_z = L_z + S_z$. Assume the nucleon has a well-defined total spin $s = \frac{1}{2}$, orbital angular momentum l, and total angular momentum j. You may also assume that $J_z = \hbar j$ in this problem.

15 **A:** Let us begin by studying the case $j = l + \frac{1}{2}$.

- A1. In the coupled basis, we are interested in the nucleon in the state $|jj\rangle$. What is this state in the uncoupled basis?
- A2. By writing μ_z in terms of J_z and S_z, calculate $\langle\mu_z\rangle$ in the state $|jj\rangle$.

15 **B:** Now consider the case $j = l - \frac{1}{2}$.

- B1. In the coupled basis, we still are interested in the nuclear state $|jj\rangle$. What is this state in the uncoupled basis?[1]
- B2. Calculate $\langle\mu_z\rangle$ in the state $|jj\rangle$. What do you find?

Our model thus predicts the magnetic moments of nuclei will take rather restricted values, and upon measuring $\langle\mu_z\rangle$ for odd (fermionic) nuclei, one finds quite good (though not perfect) agreement with this simple model.

Problem 2 (Molecular hyperfine interactions): Most molecules have net electronic spin zero, but there are can still be hyperfine interactions between a nuclear spin $\mathbf{I}$ and the orbital angular momentum $\mathbf{L}$, given by a **electric quadrupole** interaction

$$H = A\sum_{i,j=1}^{3} I_iI_j\left(L_iL_j + L_jL_i - \frac{2}{3}\delta_{ij}\mathbf{L}^2\right). \tag{2}$$

Here $A > 0$ is a constant. The components of $\mathbf{L}$ and $\mathbf{I}$ commute: $[L_i, I_j] = 0$.

10 **A:** Use the angular momentum algebra to show that we can re-write[2]

$$H = A\left(2(\mathbf{L}\cdot\mathbf{I})^2 + \mathbf{L}\cdot\mathbf{I} - \frac{2}{3}\mathbf{I}^2\mathbf{L}^2\right) \tag{3}$$

[1] *Hint:* Follow our calculation in Lecture 18.

[2] *Hint:* To write $(\mathbf{L}\cdot\mathbf{I})^2$, you need to have the i and j indices adjacent. So, write $I_iI_jL_jL_i = I_iI_j(L_iL_j - [L_i, L_j])$.

15 **B:** Suppose we have total orbital angular momentum ℓ and total nuclear spin angular momentum i_n.

B1. Find the eigenvalues of H.[3]

B2. Show that when either $i_\mathrm{n} = 0$ or $i_\mathrm{n} = \frac{1}{2}$, all eigenvalues of H are 0. In contrast, if $i_\mathrm{n} = 1$, then H has non-zero eigenvalues. You can use `Mathematica` to simplify expressions.

Unlike the hyperfine interactions in hydrogen, hyperfine interactions in molecules require a larger total nuclear spin i_n.

Problem 3 (Baryons): In particle physics, particles such as protons and neutrons are examples of a more general family of particles called **baryons**. Baryons are made up of 3 quarks. The quark is a spin-$\frac{1}{2}$ fermion that – for the purposes of our problem – carries three labels: spin s, isospin i, and color c. So a baryonic wave function will be a linear combination of kets of the form $|s_1 i_1 c_1\rangle \otimes |s_2 i_2 c_2\rangle \otimes |s_3 i_3 c_3\rangle$.

The most interesting part of the wave function for our problem is the isospin i, which labels one of two states: $|\mathrm{u}\rangle$ (up quark) and $|\mathrm{d}\rangle$ (down quark). Sometimes it is reasonable to approximate that – as there is a spin rotational symmetry in our universe – there is an *approximate* isospin-$\frac{1}{2}$ rotational symmetry, where we think of the up/down quarks as analogous to the $m = +\frac{1}{2}/-\frac{1}{2}$ states of a spin-$\frac{1}{2}$ system. The isospin states will then be grouped up to representations of isospin rotational symmetry. But because up and down states have different charge, measured by operator Q:

$$Q|\mathrm{u}\rangle = \frac{2}{3}e|\mathrm{u}\rangle, \tag{4a}$$

$$Q|\mathrm{d}\rangle = -\frac{e}{3}|\mathrm{d}\rangle, \tag{4b}$$

different combinations of up/down quarks lead to baryons of different electric charge.

10 **A:** As in Lecture 16, the wave function of the baryon must be antisymmetric under the exchange of any two quarks. Letting, for example, P_{12} be the particle exchange operator for the first two quarks:

$$P_{12}|s_1 i_1 c_1\rangle \otimes |s_2 i_2 c_2\rangle \otimes |s_3 i_3 c_3\rangle = |s_2 i_2 c_2\rangle \otimes |s_1 i_1 c_1\rangle \otimes |s_3 i_3 c_3\rangle, \tag{5}$$

we need the total baryon wave function $|\mathrm{baryon}\rangle$ to be antisymmetric under P_{12}, P_{13}, P_{23}. It turns out that due to the color symmetry of quantum chromodynamics (QCD), the baryon wave function must be written as

$$|\mathrm{baryon}\rangle = |\text{spin and isospin}\rangle \otimes |\mathrm{color}\rangle, \tag{6}$$

where $|\text{spin and isospin}\rangle$ depends on s/i, and $|\mathrm{color}\rangle$ depends on c.

A1. Does $|\mathrm{baryon}\rangle$ need to be in an even or odd representation of the ($\mathbb{Z}_2$) particle exchange P_{12}?

A2. One can show using QCD that

$$P_{12}|\mathrm{color}\rangle = P_{13}|\mathrm{color}\rangle = P_{23}|\mathrm{color}\rangle = -|\mathrm{color}\rangle. \tag{7}$$

This implies that $|\text{spin and isospin}\rangle$ must be in an even or odd representation. Which is it?

15 **B:** Now, let us think about building up $|\text{spin and isospin}\rangle$. For the moment, we'll neglect spin and focus on isospin. Suppose we have 3 isospin-$\frac{1}{2}$ quarks making up a baryon. Show that $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$.

[3] *Hint:* Follow what we did in Lecture 17.

15 **C:** This tensor product calculation is relevant for the isospin sector, and let us begin by focusing on the total isospin-$\frac{3}{2}$ sector.

C1. Explain why the state of maximal isospin (neglecting physical spin for now) is

$$\left|i = \frac{3}{2}, i_z = \frac{3}{2}\right\rangle = |\text{uuu}\rangle. \tag{8}$$

C2. Apply an isospin lowering operator, following Lecture 19, to construct the states of isospin $i = \frac{3}{2}$ but differing i_z. As a partial answer, you should find

$$\left|i = \frac{3}{2}, i_z = \frac{1}{2}\right\rangle = \frac{|\text{uud}\rangle + |\text{udu}\rangle + |\text{duu}\rangle}{\sqrt{3}}. \tag{9}$$

C3. Explain why you predict particles of charge $+2e$, $+e$, 0 and $-e$.

These baryons are called the Δ particles.

10 **D:** The wave function for a Δ baryon is of the form

$$|\Delta\rangle = \left|i = \frac{3}{2}, i_z\right\rangle \otimes |\text{spin}\rangle \otimes |\text{color}\rangle. \tag{10}$$

D1. Using previous results, explain why the wave function $|\text{spin}\rangle$, which depends on the physical spin $s_{1,2,3}$ of the three quarks, must be a fully symmetric wave function under the particle exchange operation.

D2. Conclude that the total physical spin of the Δ-particles must be $s = \frac{3}{2}$.[4]

15 **E:** Besides the Δ-particles, we still have the two isospin-$\frac{1}{2}$ representations. Do these lead to any physical particles?

E1. Ignoring physical spin again, let's look for the representations of isospin-$\frac{1}{2}$. Explain why there are two possible wave functions with total isospin $i = \frac{1}{2}$ with $i_z = \frac{1}{2}$:

$$\left|i = \frac{1}{2}, i_z = \frac{1}{2}, \text{A}\right\rangle = \frac{|\text{udu}\rangle - |\text{duu}\rangle}{\sqrt{2}}, \tag{11a}$$

$$\left|i = \frac{1}{2}, i_z = \frac{1}{2}, \text{B}\right\rangle = \frac{2|\text{uud}\rangle - |\text{duu}\rangle - |\text{udu}\rangle}{\sqrt{6}}. \tag{11b}$$

Give expressions for the $i_z = -\frac{1}{2}$ states as well.

E2. Show how to explicitly construct a baryonic spin and isospin wave function, consistent with symmetry requirements from particle exchange, in this $i = \frac{1}{2}$. You should find that there are four possible wave functions, characterized by all possible combinations of $i_z = \pm\frac{1}{2}$ and physical spin $s_z = \pm\frac{1}{2}$. These represent the spin-$\frac{1}{2}$ **proton and neutron**.

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[4] *Hint:* Think about building a wave function for 3 bosonic particles. Is there any way to do find a fully symmetric wave function involving spin up/down that does not lead to the wave functions from C2 (after replacing isospin with spin).