PHYS 4410

Quantum Mechanics 2

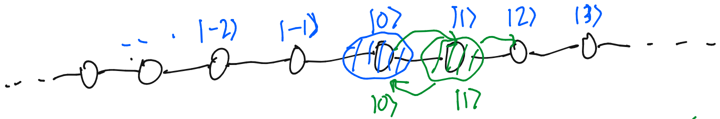
Spring 2023

Lecture 10

Discrete translation symmetry: finite chain

February 10

1 Toy model for electrons in solid:



$$H = \underset{\substack{\uparrow \\ \text{const.}}}{-\beta} \sum_{n=-\infty}^{\infty} \left[\underbrace{|n\rangle\langle n+1|}_{\text{hop left}} + \underbrace{|n+1\rangle\langle n|}_{\text{hop right}} \right]$$

single electron

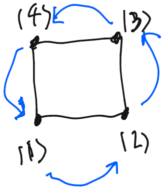
$$H|n\rangle = -\beta\left[|n-1\rangle + |n+1\rangle\right]$$

$$\delta_{n+1,n'} = \begin{cases} 1 & n+1=n' \\ 0 & \text{else} \end{cases}.$$

$$\left[\sum_{n=-\infty}^{\infty} |n\rangle\langle n+1|\right]|n'\rangle = \sum_{n=-\infty}^{\infty} |n\rangle \underbrace{\langle n+1|n'\rangle}_{\delta_{n+1,n'}} = |n'-1\rangle.$$

How to diagonalize H?

2 A) look for "simpler" problem.



$$H = -\beta \sum_{n=1}^{4} \Big[|n\rangle\langle n+1| + |n+1\rangle\langle n| \Big]$$

identify $\underline{|4+1\rangle = |1\rangle}$.

B) look for a symmetry.

$$R = \sum_{n=1}^{4} |n+1\rangle\langle n| \quad \text{or} \quad R|n\rangle = |n+1\rangle$$

Claim: R is a symmetry (transformation).

$[H, R] = 0$ ⟵ (discrete) translation symmetry

$$\underbrace{H R |n\rangle}_{H|n+1\rangle} \stackrel{?}{=} R H |n\rangle \quad (\text{for any } |n\rangle)$$

$$H|n+1\rangle \qquad\qquad R\big[-\beta|n-1\rangle - \beta|n+1\rangle\big]$$

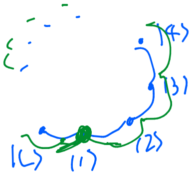
$$= -\beta|n+2\rangle - \beta|n\rangle \qquad\qquad = -\beta|n\rangle - \beta|n+2\rangle$$

$\boxed{3}$ Recall Thm: $[H, R] = 0$, basis where H, R both diagonal.

Can we find eigenvector of R?

$R|n\rangle = |n+1\rangle$ $\quad [R|L\rangle = |1\rangle]$.

L sites:



If $R^k|n\rangle = |n+k\rangle$

$R^L = \mathbb{1}$ (identity).

And $\{\mathbb{1}, R, R^2, \ldots, R^{L-1}\}$ form symmetry group. $(\mathbb{Z}_L)$

$R|\lambda\rangle = \lambda|\lambda\rangle$, then $R^L|\lambda\rangle = \lambda^L|\lambda\rangle$ So $\lambda^L = 1$.

$\mathbb{1}|\lambda\rangle = 1 \cdot |\lambda\rangle$

Solution: "roots of unity" $\lambda = e^{\frac{2\pi i}{L} \cdot j}$ $\quad j = 1, \ldots, L$

$L\theta = 2\pi n$ $\quad r^L = 1$ $\quad \leftarrow \quad \lambda = (re^{i\theta})^L = 1$.

$\boxed{4}$ What state $|\bar{k}\rangle$ obeys: $\underbrace{R|\bar{k}\rangle = e^{-2\pi i/L \cdot k}|\bar{k}\rangle}_{\langle \ell+1|}$

Recall: $\underline{R = \sum_n |n+1\rangle\langle n|}$

$$\underbrace{\langle \ell+1|R}|\bar{k}\rangle = e^{-2\pi i/L \cdot k}\langle \ell+1|\bar{k}\rangle$$

$$\langle \ell|\bar{k}\rangle = e^{-2\pi i/L \cdot k}\langle \ell+1|\bar{k}\rangle$$

$$\langle 2|\bar{k}\rangle = \underline{\langle 1|\bar{k}\rangle} \cdot e^{2\pi i k/L}$$

$$\langle 3|\bar{k}\rangle = \langle 2|\bar{k}\rangle \cdot e^{2\pi i k/L}$$

$$\vdots$$

$$|\bar{k}\rangle = \frac{1}{\sqrt{L}}\sum_{\ell=1}^{L} |\ell\rangle \cdot e^{\left[\left(\frac{2\pi i k}{L}\right)\cdot \ell\right]} \quad \Bigg\}$$ discrete Fourier transform.

$\boxed{5}$ Recap: symmetry transform R: $[H,R]=0$.

here: $R^L = \mathbb{1}$. [$\mathbb{Z}_L$ symmetry]

The eigenvalues of R: $\lambda = e^{-2\pi i/L \cdot k}$ $k=1,\dots,L$

e-vector: $\underbrace{|\bar{k}\rangle}_{\text{"momentum" space}} = \sum \frac{1}{\sqrt{L}} e^{+2\pi i k/L \cdot \ell} \underbrace{|\ell\rangle}_{\text{position space}}$

Math jargon: each eigenvalue λ $\exists$ "eigenspace" $|\bar{k}\rangle$

(irrep =) irreducible representation

$R|\bar{k}\rangle$ → other state in irrep

Schur's Lemma: eigen-vec of H organize into irreps ($|\bar{k}\rangle$)

and e-vals same in each (copy of) irrep.

$\boxed{6}$

$$H = -\beta \sum_{n=1}^{L} \left[|n\rangle\langle n+1| + |n+1\rangle\langle n| \right]$$

Since $[H, \hat{R}] = 0$, $|\tilde{k}\rangle$ is eigenvector of H.

$$H|\tilde{k}\rangle = \frac{1}{\sqrt{L}} \sum_n (e^{2\pi i k/L})^n \underbrace{H|n\rangle}_{-\beta|n-1\rangle - \beta|n+1\rangle}$$

$$= -\frac{\beta}{\sqrt{L}} \sum_n \Big[\underbrace{(e^{2\pi i k/L})^{n-1+1}}_{e^{2\pi i k/L} \cdot e^{2\pi i k/L \cdot n'}} \underbrace{|n-1\rangle}_{n' = n-1} + (e^{2\pi i k/L})^{n+1-1} |n+1\rangle \Big]$$

$$= -\beta \Big[e^{2\pi i k/L} \underbrace{\Big(\frac{1}{\sqrt{L}} \sum_{n'} e^{2\pi i k n'/L} |n'\rangle \Big)}_{|\tilde{k}\rangle} + \cdots \Big]$$

$$2\cos\theta = e^{+i\theta} + e^{-i\theta}$$

$\boxed{k = 1, \ldots, L}$

$$= -\beta \left(e^{2\pi i k/L} + e^{-2\pi i k/L} \right) |\tilde{k}\rangle = -2\beta \cos\left(\frac{2\pi k}{L}\right) |\tilde{k}\rangle$$

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