PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 13

Continuous translation symmetry

February 20

$\boxed{1}$ A symmetry group G is a set of unitary matrices $G = \{U\}$ and $[H, U]$ for any $U \in G$. and $\underbrace{U_1 U_2}_{\in G} \in G$

Sometimes G can be continuous (Lie) group.

Example: translation symmetry in free particle:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} = \frac{p^2}{2m} \qquad U(a)\,\psi(x) = \psi(x+a)$$

cont.

Claim: $\{U(a)$ for any real $a\}$ is a group.

$$U(a)U(b)\psi(x) = U(a)\psi(x+b) = \psi(x+[b+a]) = U(b+a)\psi(x)$$

$$U(a)U(b) = U(a+b) = U(b)U(a)$$

Group multiplication is commutative (order doesn't matter)

Abelian group

$\boxed{2}$ $\left([H, U(a)] = 0\right)\psi(x) \longrightarrow H\,U(a)\,\psi - U(a)\,H\psi = 0.$

$$H\,U(a)\,\psi(x) = H\,\psi(x+a) = -\frac{\hbar^2}{2m}\underbrace{\frac{d^2}{dx^2}\psi(x+a)}_{\frac{d}{dx}\left[\frac{d(x+a)}{dx}\psi'(x+a)\right] = \psi''(x+a)}$$

$$= U(a)\left(H\psi(x)\right)$$

Claim: Let $p = -i\hbar\frac{d}{dx}$. Claim: $U(a) = e^{ipa/\hbar}$.

~~Matrix~~ operator exponential: $e^A = \sum_{n=0}^{\infty}\frac{A^n}{n!}$. if A is a number or matrix.

$$= 1 + A + \frac{1}{2}AA + \cdots$$

$$U(a)\,\psi(x) = \psi(x+a) = \psi(x) + \psi'(x)\,a + \frac{1}{2}\psi''(x)a^2 + \cdots$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}a^n\left.\frac{d^n\psi}{dx^n}\right|_x = \left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(a\frac{d}{dx}\right)^n\right]\psi(x) = e^{a\frac{d}{dx}}\psi$$

$= ipa/\hbar$

$\boxed{3}$

Why is $U(a)$ unitary?

$\varphi(x) = U(a)\,\psi(x) = \psi(x+a)$

$\langle \varphi | \varphi \rangle = \langle \psi | \psi \rangle$?

$x' = x + a$

$$\int_{-\infty}^{\infty} dx\, |\varphi(x)|^2 = \int_{-\infty}^{\infty} dx\, |\psi(x+a)|^2 = \int_{-\infty}^{\infty} dx'\, |\psi(x')|^2 = \langle \psi | \psi \rangle.$$

Proposition: if B is Hermitian, e^{iB} is unitary.

$U(a)$: $B = p(a/\hbar)$, and momentum p is Hermitian.

Check: $U(a)^{-1} = U(a)^{\dagger} = U(-a)$

$\boxed{4}$ $U(a) = e^{ipa/\hbar}$: (p) momentum generates translation symmetry.

Note: Instead of checking $[H, U(a)] = 0$ for all a...

can check $[H, p] = 0$. Take $f(p) = e^{\frac{ia}{\hbar} \cdot p}$

Reason: If $[H, p] = 0$ then $[H, f(p)] = 0$ for any function f.

Note: p & f(p) diagonal in same basis. $[p, f(p)] = 0$.

Show $[H, U(a)] = 0 \rightarrow [H, p] = 0$.

If $a \to 0$: $U(a) \approx 1 + i\frac{a}{\hbar} \cdot p + \ldots$

$$[H, 1] + i\frac{a}{\hbar}[H, p] + \ldots = 0.$$

Why does $[H, p] = 0$?

$$H = \frac{1}{2m} p^2 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}$$

since H is a function of p.

$\boxed{5}$ What are eigenfunctions of p?

$p\,\psi(x) = \underset{\text{const. eigenval.}}{p_0}\,\psi(x)$

$-i\hbar \dfrac{d\psi}{dx} = p_0 \psi$

Solve linear equation: $\psi(x) = \underbrace{e^{i p_0/\hbar \cdot x}}_{\text{plane wave fixed mom. } p_0}$.

Since $[H, p] = 0$, share eigenvector/function

$$H\, e^{i p_0 x/\hbar} = E\, e^{i p_0 x/\hbar} \qquad \text{with } E = \frac{p_0^2}{2m}$$

In general $f(p)|p_0\rangle = f(p_0)|p_0\rangle$.

Jargon: $e^{i p_0 x/\hbar}$, labeled by p_0, irreducible representation of translation.

single function $\longrightarrow$ 1-dimensional irrep.

6 Angular coordinates

ϕ is the same $\phi + 2\pi$

$\phi \sim \phi + 2\pi$.

The "conjugate momentum" is L_z: "$[\phi, L_z]$" $= i\hbar$

$L_z = -i\hbar \frac{d}{d\phi}$.

Translation symmetry: $\{U(\alpha)$ for any $\alpha\}$: $U(\alpha + 2\pi) = U(\alpha)$

$U(\alpha) = e^{i\alpha L_z/\hbar}$.

If $1 = U(0) = U(2\pi)$, then: $1 = e^{2\pi i L_z/\hbar}$

So if $L_z = m\hbar$ ← eigenvalue of L_z: $e^{2\pi i m} = 1$.

$m = 0, \pm 1, \pm 2, \ldots$ is an integer.