PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 15

Angular momentum algebra

February 24

$\boxed{1}$ Rotation group generated by L_x, L_y, L_z:

$$\left.\begin{aligned} [L_x, L_y] &= i\hbar L_z \\ [L_y, L_z] &= i\hbar L_x \\ [L_z, L_x] &= i\hbar L_y \end{aligned}\right\}$$

QM: have orbital ($\vec{L}$) or spin ($\vec{S}$) angular momentum: use $\vec{J} = \vec{L} + \vec{S}$

$[J_z, J_x] = i\hbar J_y$.

Goal: if $[H, J_x] = [H, J_y] = [H, J_z] = 0$ ... what are possible representations realized?

Answer:

$$\left.\begin{aligned} &\vec{J}^2 |jm\rangle = \hbar^2 j(j+1)|jm\rangle \\ &\qquad j = 0, 1/2, 1, 3/2, \ldots \\ &J_z |jm\rangle = \hbar m |jm\rangle \\ &\qquad m = -j, 1-j, \ldots, j \end{aligned}\right\}$$

for each allowed j: $2j+1$ allowed m. irr. rep.

$\boxed{2}$ Claim #1: Simultaneously diag. $\vec{J}^2$ & J_z:

$$\vec{J}^2 = J_x^2 + J_y^2 + J_z^2 .$$

$$[AB, C] = A[B,C] + [A,C]B$$

$$\begin{aligned}
[\vec{J}^2, J_z] &= 0 \\
&= [J_x^2 + J_y^2 + J_z^2, J_z] \\
&= J_x [J_x, J_z] + [J_x, J_z] J_x + \cdots \qquad \cancel{[J_z^2, J_z]} \to 0 \\
&= J_x(-i\hbar J_y) + (-i\hbar J_y) J_x + \cdots \\
&= 0
\end{aligned}$$

Choice of J_z is arbitrary.

Could have $[\vec{J}^2, J_x] = 0 \ldots$

- if J_z is chosen to be diagonal, J_x / J_y not diagonal.

3 Claim #2: raising and lower operators:

$$J_+ = J_x + iJ_y$$

$$J_- = J_x - iJ_y$$

$$J_+^\dagger = J_-$$

$$[J_+, J_z] = [J_x + iJ_y, J_z]$$

$$= -i\hbar J_y + i(i\hbar J_x)$$

$$= -\hbar(iJ_y + J_x) = -\hbar J_+ .$$

Also: $[J_-, J_z] = \hbar J_-$; $[J_+, J_-] = 2\hbar J_z$; $[\vec{J}^2, J_\pm] = 0$.

<u>Claim #3</u>: Suppose $J_z|m\rangle = \hbar m|m\rangle$

Then: $J_z(J_\pm|m\rangle) = \hbar(m\pm1)(J_\pm|m\rangle)$

Check:

$$J_z J_+|m\rangle = (J_+J_z + [J_z, J_+])|m\rangle$$

$$= (J_+J_z - (-\hbar J_+))|m\rangle$$

$$= (\hbar m + \hbar)\, J_+|m\rangle$$

eigenvalue $\underline{\hbar(m+1)}$

4 Claim #4: m cannot be arbitrarily large:

$$\vec{J}^2 = J_x^2 + J_y^2 + J_z^2.$$

$$= (J_x + iJ_y)(J_x - iJ_y) - iJ_yJ_x + iJ_xJ_y + J_z^2$$

$$= J_+J_- + i(i\hbar J_z) + J_z^2 = \underline{J_+J_- + J_z(J_z - \hbar)}$$

if $|\varphi\rangle = J_z|m\rangle$, then $\langle\varphi|\varphi\rangle \geq 0$.

$$\langle m|\vec{J}^2|m\rangle \geq 0.$$

$$= \underbrace{(\langle m|J_x)(J_x|m\rangle)}_{\geq 0.} + (\langle m|J_y)(J_y|m\rangle) + (\langle m|J_z)J_z|m\rangle$$

If m is too large... $\vec{J}^2|m\rangle = \lambda|m\rangle$:

$$\lambda\langle m|m\rangle = \underbrace{\langle m|J_x^2 + J_y^2|m\rangle}_{\geq 0} + \underbrace{\langle m|J_z^2|m\rangle}_{(\hbar m)^2\langle m|m\rangle}$$

$\lambda \geq \hbar^2 m(m-1)$

&

Saturated if $J_-|m\rangle = 0$

same:

$$\lambda\langle m|m\rangle = \underbrace{\langle m|J_-^\dagger J_-|m\rangle}_{\geq 0} + \hbar^2 m(m-1)\langle m|m\rangle$$

$\boxed{5}$ Doing similar calculation:

$\lambda|m\rangle = \hbar^2 m(m+1)|m\rangle$ when $\underbrace{J_+|m\rangle = 0.}$ must happen.

Conclude: must be a largest m: $|m_{max}\rangle$

$$\lambda = \hbar^2 m_{max}(m_{max}+1)$$

and a smallest m: $|m_{min}\rangle$

$$\lambda = \hbar^2 m_{min}(m_{min}-1)$$

And... $J_+^k |m_{min}\rangle = \# |m_{max}\rangle$ for some integer k.

$k + m_{min} = m_{max}$. $\underline{(k=0,1,2,3,\ldots)}$

Solution: $m_{min} = -k/2$ & $m_{max} = +k/2$.

Call $k = 2j$ $\rightarrow$ $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ $\lambda = \hbar^2 j(j+1)$

$m = -j, 1-j, \ldots, +j$.

$\vec{J}^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle$ and $J_z|jm\rangle = \hbar m|jm\rangle$.

$\boxed{6}$ Choose: $\langle jm|jm\rangle = 1.$ $\quad J_\pm|jm\rangle = ?$

$$\langle jm|\vec{J}^2|jm\rangle = \langle jm|J_+J_- + J_z^2 - \hbar J_z|jm\rangle$$

$$\hbar^2 j(j+1) = \left[\langle jm|J_+\right]\left[J_-|jm\rangle\right] + \hbar^2 m(m-1)\,\cancel{\langle jm|jm\rangle}^{\,1}$$

$$J_-|jm\rangle = c\,|j,m-1\rangle \quad \hookrightarrow \langle j,m-1|c^* c|j,m-1\rangle = c^2$$

$$c^2 = \hbar^2[j(j+1) - m(m-1)], \quad J_-|jm\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\;|j,m-1\rangle.$$

Similarly: $J_+|jm\rangle = \hbar\sqrt{j(j+1)-m(m+1)}\;|j,m+1\rangle.$

Today: classified all irreps of rotation group: SU(2).

- $j = 0, 1/2, 1, 3/2, \ldots$ $\quad$ <u>spin</u>
- $j = 1/2, 3/2, \ldots \rightarrow$ fermions
  irreps not consistent w/ spherical coords.
- $j = 0, 1, 2, \ldots \rightarrow$ bosonic. $\quad$ [$Y_{\ell m}$: $j \to \ell$, orbital a.m.]