PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 16

Symmetry in two-particle systems

February 27

$\boxed{1}$ Review angular momentum:

$[J_x, J_y] = i\hbar J_z \qquad [J_y, J_z] = i\hbar J_x \qquad [J_z, J_x] = i\hbar J_y.$

Simultaneous diagonalize: $J_x^2 + J_y^2 + J_z^2 = \vec{J}^2$ & J_z.

$\vec{J}^2 |jm\rangle = \hbar^2 j(j+1) |jm\rangle \qquad J_z |jm\rangle = \hbar m |jm\rangle$

$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \qquad m = -j, 1-j, \ldots, j$

$\Big\}$ j labels representation ($2j+1$ states).

Raising/lowering operators:

$J_+ = J_x + i J_y \qquad J_+ |jm\rangle = \hbar \sqrt{j(j+1) - m(m+1)}\, |j, m+1\rangle$

$J_- = J_x - i J_y = J_+^\dagger \qquad J_- |jm\rangle = \hbar \sqrt{j(j+1) - m(m-1)}\, |j, m-1\rangle$

$$J_x = \frac{J_+ + J_-}{2}$$

$$J_y = \frac{J_+ - J_-}{2i}$$

$\boxed{2}$ Find J_x, J_y, J_z for $j = 1/2$.

$J_+ |\uparrow\rangle = 0$

$J_+ |\downarrow\rangle = \, ? \, |\uparrow\rangle$

Work in basis w/ J_z diagonal.

$$J_z = \begin{array}{cc} |\uparrow\rangle & |\downarrow\rangle \\ \end{array}$$
$$J_z = \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix} \qquad \left[\text{e.g. } |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$

$$J_+ = \begin{pmatrix} 0 & \hbar \\ 0 & 0 \end{pmatrix}$$

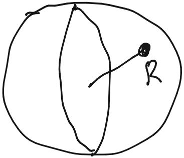
Figure out J_+: $$J_+ \underbrace{\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle}^{|\downarrow\rangle} = \hbar \sqrt{\underbrace{\tfrac{1}{2}}_{j}\left(\tfrac{1}{2}+1\right) - \underbrace{\left(-\tfrac{1}{2}\right)}_{m}\left(-\tfrac{1}{2}+1\right)} \; \left|\tfrac{1}{2}\; \tfrac{1}{2}\right\rangle$$
$$= \hbar \left|\tfrac{1}{2}\, \tfrac{1}{2}\right\rangle$$

$$J_- = J_+^\dagger = \begin{pmatrix} 0 & 0 \\ \hbar & 0 \end{pmatrix}.$$

Use: $$J_x = \frac{J_+ + J_-}{2} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$J_y = \frac{J_+ - J_-}{2i} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \leftarrow \frac{1}{i} = -i$$

$\boxed{3}$



Particle on sphere of radius R, mass M.

$$H = \frac{\vec{L}^2}{2I} \qquad I = MR^2 ; \qquad \vec{L} = \text{orbital angular momentum.}$$

What are e-values/states of H?

$$E_l = \frac{\hbar^2 l(l+1)}{2MR^2}, \quad l = 0, 1, 2, \dots$$

e-states $|lm\rangle$

$m = -l, 1-l, \dots, l$

What if: $H = A\vec{L}^2 + BL_z$? $\qquad [\vec{L}^2, L_z] = 0$

$$H|lm\rangle = \left[A\hbar^2 l(l+1) + B\hbar m\right]|lm\rangle$$

What if: $H = A\vec{L}^2 + BL_y$? $\qquad [\vec{L}^2, L_y] = 0.$

energy levels same

$\boxed{4}$ Next goal: Lec 17-19: multiple particles w/ rotational symmetry.

Toy example: 2 particles in harmonic oscillator:

$$H = \left[\frac{p_1^2}{2m} + \frac{1}{2}m\omega_1^2 x_1^2\right] + \left[\frac{p_2^2}{2m} + \frac{1}{2}m\omega_2^2 x_2^2\right]$$

$H|n_1 n_2\rangle = \hbar\omega(n_1+n_2+1)|n_1 n_2\rangle$.

$(x_1 \to -x_1)$ $(x_2 \to -x_2)$

Parity symmetry: $P = P_1 P_2 = P_{single} \otimes P_{single} \ldots$

$(x_1, x_2) \to (-x_1, -x_2)$

Recall: $P_{single}|n\rangle = (-1)^n |n\rangle$.

$$P|n_1 n_2\rangle = (-1)^{n_1+n_2}|n_1 n_2\rangle$$

$$= P_{single}|n_1\rangle \otimes P_{single}|n_2\rangle = (-1)^{n_1}|n_1\rangle \otimes (-1)^{n_2}|n_2\rangle$$

$\boxed{5}$ So $P|n_1,n_2\rangle = (-1)^{n_1+n_2}|n_1 n_2\rangle$ also eigenvector of P.

$f(x_1)\,g(x_2)$ $\tilde{f}(x_1,x_2)$

n_1 even, n_2 even $\quad$ n_1+n_2 even

$+1 \cdot +1 = +1$

even $\otimes$ even = even

n_1 even, n_2 odd $\quad$ n_1+n_2 odd

$+1 \cdot -1 = -1$

even $\otimes$ odd = odd

odd $\otimes$ even = odd

n_1 odd $\quad$ n_2 even : n_1+n_2 odd

n_1 odd $\quad$ n_2 odd : n_1+n_2 even.

odd $\otimes$ odd = even

representation of parity $\quad$ representation.

Tensor products of representation $\rightarrow$ decomposing into representation

$\boxed{6}$ Application: particle exchange symmetry of composite:

Hydrogen atom: "$|H\rangle = \underset{\text{proton}}{|p\rangle} \otimes \underset{\text{electron}}{|e\rangle}$". p, e: spin-$\frac{1}{2}$ fermions

Would $|H\rangle$ behave as a boson or fermion?

Particle exchange: $P\,|p_1 e_1\, p_2 e_2\rangle = |p_2 e_2\, p_1 e_1\rangle$

Abusing notation...

$$\frac{|p_1 p_2\rangle - |p_2 p_1\rangle}{\sqrt{2}} = |p's\rangle \qquad P\,|p's\rangle = -|p's\rangle$$

$$P\,|e's\rangle = -|e's\rangle$$

$$P\,|H's\rangle = P\big(|p's\rangle \otimes |e's\rangle\big) = [\overset{+1}{\cancel{(-1)^2}}]\,|H's\rangle$$

: H is a boson.

$|H's\rangle$ is in odd $\otimes$ odd = even representation of parity

if we had 3 fermions: (odd $\otimes$ odd) $\otimes$ odd = even $\otimes$ odd = odd.