PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 17

The hyperfine interaction

March 1

$\boxed{1}$ Hyperfine interaction btween electron ($\vec{S}$) / proton ($\vec{I}$) spin-1/2

$$H = A\,\vec{S}\cdot\vec{I}$$

$$[S_x, I_x] = 0$$
$$[S_x, I_y] = 0.$$

What are e-values/e-vectors?

Problem has rotational symmetry

Hilbert space: $(\text{spin } \tfrac{1}{2})_e \otimes (\text{spin } \tfrac{1}{2})_p = \frac{1}{2}\otimes\frac{1}{2}$

$$|\uparrow\uparrow\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |\uparrow\downarrow\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |\downarrow\uparrow\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |\downarrow\downarrow\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

$$S_x I_x = S_x \otimes I_x$$

Spin matrices for a single spin:

$$S_x = \begin{array}{c} \begin{array}{cc} |\uparrow\rangle & |\downarrow\rangle \end{array} \\ \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \end{array} \frac{\hbar}{2} \qquad S_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\frac{\hbar}{2} \qquad S_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\frac{\hbar}{2}.$$

$\boxed{2}$ Solution #1: brute

$$H = A\left[S_x I_x + S_y I_y + S_z I_z\right]$$

$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ e.g., so

$$S_x I_x |\uparrow\uparrow\rangle = \left(\frac{\hbar}{2}\right)^2 |\downarrow\downarrow\rangle$$

$$S_x I_x |\uparrow\downarrow\rangle = \left(\frac{\hbar}{2}\right)^2 |\downarrow\uparrow\rangle$$

$\vdots$

$$S_x I_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\left(\frac{\hbar}{2}\right)^2$$

Since $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$S_z I_z = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Repeat for:

$$S_y I_y = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Combine terms:

$$H = A\left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\boxed{3}$

$$H = A\left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What are e-values/e-vectors.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad H|\uparrow\uparrow\rangle = A\frac{\hbar^2}{4}|\uparrow\uparrow\rangle$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad H|\downarrow\downarrow\rangle = A\frac{\hbar^2}{4}|\downarrow\downarrow\rangle.$$

Diagonalize : $\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} = -1 \cdot \mathbb{1} + \underbrace{2\sigma_x}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$H|e\rangle = (2-1)\frac{A\hbar^2}{4}|e\rangle \qquad +1 : \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \qquad -1 : \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$|e\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \qquad |o\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}.$$

$$H|o\rangle = (-2-1)\frac{A\hbar^2}{4}|o\rangle$$

Summary:

$H = A\hbar^2/4 :\ |e\rangle,\ |\uparrow\uparrow\rangle,\ |\downarrow\downarrow\rangle$

$H = -3A\frac{\hbar^2}{4} :\ |o\rangle$

$\boxed{4}$ $$H = A\frac{\hbar^2}{4}\underbrace{\left[|e\rangle\langle e| + |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|\right]}_{\text{triple degeneracy. why? spin 1}} - \underbrace{\frac{3A\hbar^2}{4}|o\rangle\langle o|}_{\text{spin 0}}.$$

Symmetry: rotation. Since $H = A\vec{S}\cdot\vec{I}$, although $\vec{S}$ and $\vec{I}$ aren't rotation invariant, dot product invariant under rotation.

For any unitary U implementing rotation $[H, U] = 0$.

$$\underbrace{\frac{1}{2}\otimes\frac{1}{2}}_{\text{tensor product: 2 spin-1/2 particles}} = \underbrace{0 \oplus 1}_{\text{direct sum of spin-0 DOF + spin-1 DOF.}}$$

$\boxed{5}$ let $\vec{J} = \vec{S} + \vec{I}$ [$\vec{J}$ generated rotations $[H, \vec{J}] = 0$].

Try to diagonalize $\vec{J}^2$ & J_z: $J_z|e\rangle = 0|e\rangle$.

$|\uparrow\uparrow\rangle$ spin 1 $|e\rangle$ or $|0\rangle$ spin 0 $|\downarrow\downarrow\rangle$

$J_z = \frac{\hbar}{2} + \frac{\hbar}{2} = \hbar$ $J_z = 0$ $J_z = -\hbar$.

How do we identify which of $j = 0, \cancel{\frac{1}{2}}, 1, \ldots$ we have?

- any half-integer has $J_z = 1/2$ e.g.
- $j = 2, 3 \ldots$ forbidden b/c no state has $J_z = 2\hbar$.
- irrep of spin j has dimension $2j+1$
- so only include $j=0$ or 1.
- $J_z = \hbar$ implies $j=1$ included...

  4 total states: $0 \oplus 1$

  $= 1 + 3$

$\boxed{6}$ Solution #2: "angular momentum addition"

$$\boxed{\tfrac{1}{2}\otimes\tfrac{1}{2} = 0\oplus 1}$$

$$\vec{I}\cdot\vec{S} + \vec{S}\cdot\vec{I} = 2\vec{S}\cdot\vec{I}$$

let $\vec{J} = \vec{S} + \vec{I}$.

$$H = A\vec{S}\cdot\vec{I} = \frac{1}{2}A\left[(\vec{S}+\vec{I})\cdot(\vec{S}+\vec{I}) - \vec{S}^2 - \vec{I}^2\right]$$

$$= \frac{A}{2}\left[\vec{J}^2 - \vec{S}^2 - \vec{I}^2\right]$$

e/p both spin-1/2: $\vec{S}^2 = \vec{I}^2 = \hbar^2\cdot\frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{4}\hbar^2$.

$\vec{J}^2$ be assoc. $j=0$ or $j=1$: $\vec{J}^2 = \begin{cases} \hbar^2\cdot 0(0+1) & j=0 \\ \hbar^2\cdot 1(1+1) & j=1 \end{cases}$

$$H = \frac{A}{2}\left[\hbar^2 j(j+1) - 2\cdot\frac{3}{4}\hbar^2\right] = \begin{cases} \frac{A}{4}\hbar^2 & j=1 \\ -\frac{3}{4}A\hbar^2 & j=0 \end{cases}$$

$j=1$: $m=1$, $m=0$, $m=-1$

$\frac{A}{4}\hbar^2$: $\{|\uparrow\uparrow\rangle, |e\rangle, |\downarrow\downarrow\rangle\}$

$j=m=0$:

$-\frac{3}{4}A\hbar^2$: $|o\rangle$