PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 18

Angular momentum addition

March 3

$\boxed{1}$ Recall: $H = A \underset{\text{spin-}1/2\ e^-}{\vec{S}} \cdot \underset{\text{spin-}1/2\ p.}{\vec{I}}$

$\vec{J} = \vec{S} + \vec{I}$

$[H, \vec{J}] = 0.$

H can be diagonalized w/ $\vec{J}^2$, J_z.

eigenvalues/vectors:

$$E = \frac{A\hbar^2}{4}: \quad \left\{ \begin{array}{l} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{array} \right. \qquad \left. \begin{array}{l} |j=1, m=1\rangle \\ |j=1, m=0\rangle \\ |j=1, m=-1\rangle \end{array} \right\}$$

$\vec{J}^2 = \hbar^2 j(j+1)$

$J_z = \hbar m$

"triplet"

$$\underbrace{E = -\frac{3A\hbar^2}{4}: \quad \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)}_{\text{"uncoupled basis"}} \qquad \underbrace{|j=0, m=0\rangle}_{\text{"coupled basis"}}$$

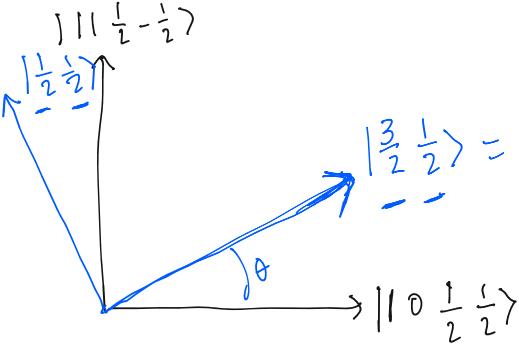
"singlet"

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

Today's goal:

$$\underbrace{j_1 \otimes j_2}_{\text{“uncoupled basis”}} = \underbrace{|j_1-j_2| \oplus (|j_1-j_2|+1) \oplus \cdots \oplus (j_1+j_2)}_{\text{“coupled basis”}}$$

$|j_1 m_1 j_2 m_2\rangle$ $\qquad$ $|jm\rangle$



$$|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \cos\theta\,|1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \sin\theta\,|1\,1\,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$\boxed{3}$ What's $j_1 \otimes j_2$? Organize by $J_z = J_{1z} + J_{2z}$

$(m) \quad (m_1) \quad (m_2) \quad \times \hbar$

max m_1 max m_2 $\quad j \cancel{\neq} = j_1 + j_2$

$m = \underline{j_1 + j_2}$: $|j_1 \overset{m_1}{j_1} j_2 \overset{m_2}{j_2}\rangle$ [since no state w/ $m = j_1 + j_2 + 1$, $j \neq j_1 + j_2 + 1$]) 1 state. $\quad 1 - 1 = 0$

$m = \underline{j_1 + j_2 - 1}$: $|j_1 (j_1 - 1) j_2 j_2\rangle$ or $|j_1 j_1 j_2 (j_2 - 1)\rangle$ 2 states $\quad -1 = 1 - 1 = 0$

$m = j_1 + j_2 - 2$: $|j_1 (j_1 - 2) j_2 j_2\rangle$ or $|j_1 (j_1 - 1) j_2 (j_2 - 1)\rangle$ or $|j_1 j_1 j_2 (j_2 - 2)\rangle$ 3 states $\quad -1 = 2 - 1 = 0$

$\vdots$

$m = -j_1 - j_2 + 1$ 0 states

$m = -j_1 - j_2$: $|j_1 (-j_1) j_2 (-j_2)\rangle$ 1 states $\quad -1 = 0$

$$j_1 \otimes j_2 = \underset{j=}{(j_1 + j_2)} \oplus \underset{j=}{(j_1 + j_2 - 1)} \oplus \cdots \oplus |j_1 - j_2|$$

$$\boxed{4} \quad \underbrace{\underset{j_1}{1} \otimes \underset{j_2}{1}}_{\text{uncoupled.}} = \underset{1+1}{2} \oplus \underset{1+1-1}{1} \oplus \underset{|1-1|}{\underline{0}} \Big\} \text{ coupled basis}$$

$\underset{\vec{a}}{1} \otimes \underset{\vec{b}}{1}$: spin 0: $\vec{a} \cdot \vec{b}$ $\qquad \mathrm{tr}(\overbrace{a_i b_j}^{M_{ij}})$

spin 1: $\vec{a} \times \vec{b}$ $\qquad a_i b_j - a_j b_i$

spin 2: traceless symmetric matrix:

$$M_{ij} = a_i b_j + a_j b_i - \frac{2}{3} \delta_{ij} \vec{a} \cdot \vec{b} .$$

5 independent components.

$\boxed{5}$ 2 particles w/ $j_1 = j_2 = 1$.

$H = A\,\vec{J}_1 \cdot \vec{J}_2$ (A some constant).

$\vec{J} = \vec{J}_1 + \vec{J}_2$. Know $\vec{J}_1^{\,2} = \vec{J}_2^{\,2} = \hbar^2 1(1+1) = 2\hbar^2$.

$$H = \frac{1}{2} A\left[(\vec{J}_1 + \vec{J}_2)^2 - \vec{J}_1^{\,2} - \vec{J}_2^{\,2}\right]$$

$$= \frac{A}{2}\enclose{circle}{\vec{J}^2} - A \cdot 2\hbar^2$$

$\vec{J}^2 = \hbar^2 j(j+1)$. $j = 0, 1, 2$ since $1 \otimes 1 = 2 \oplus 1 \oplus 0$.

$j = 0$	$j = 1$	$j = 2$
$E = -2A\hbar^2$	$E = -A\hbar^2$	$E = +A\hbar^2$
degeneracy 1	degeneracy $2j+1 = 3$	degeneracy 5

$\boxed{6}$ 3 particles of spin 1.

$$\vec K^2 = \begin{cases} \hbar^2\cdot 0 & k=0 \\ \hbar^2\cdot 1(1+1) & 1 \\ \hbar^2\cdot 2(2+1) & 2 \end{cases}$$

$$1\otimes 1\otimes 1 \overset{?}{=} (1\otimes 1)\otimes 1$$

$$|\,(|m_1\rangle|m_2\rangle)\,|m_3\rangle = (0\oplus 1\oplus 2)\otimes 1$$

(27 total states)

$$= (\underset{k}{0}\otimes \underset{j_3}{1})\oplus(1\otimes 1)\oplus(2\otimes 1)$$

$\vec K = \vec J_1 + \vec J_2,\ \vec J_3$

$$= \underline{1}\oplus(2\oplus\underline{1}\oplus\underline{0})\oplus(3\oplus 2\oplus\underline{1})$$

"3 combinations" w/ $\vec J^2 = (\vec J_1+\vec J_2+\vec J_3)^2$

$= \hbar^2\, 1(1+1)$

$\vec a \quad \vec b \quad \vec c$

$1\otimes 1\otimes 1$: $(\vec a\cdot\vec b)\,\vec c$

$(\vec a\cdot\vec c)\,\vec b$

$(\vec b\cdot\vec c)\,\vec a$

$\vec a\ \vec b\ \vec c$ multiply to get a number?

$\vec a\cdot(\vec b\times\vec c)$