PHYS 4410
Quantum Mechanics 2
Spring 2023

Lecture 19

Clebsch-Gordan coefficients

March 6

$\boxed{1}$ Review: angular momentum addition
spin j_1 particle interacting w/ j_2 spin $(H = \vec{J}_1 \cdot \vec{J}_2)$

$$\underbrace{j_1 \otimes j_2}_{\text{uncoupled}} = \underbrace{|j_1 - j_2| \oplus (|j_1 - j_2| + 1) \oplus \cdots \oplus (j_1 + j_2)}_{\text{coupled}}$$

$|j_1 m_1 j_2 m_2\rangle \longleftrightarrow |j m\rangle$ (allowed val of j above)

$m_1 = -j_1, \ldots, j_1$
$m_2 = -j_2, \ldots, j_2$

Goal: find basis transformation

Example: for $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$

$$|1\,1\rangle = |\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\rangle$$

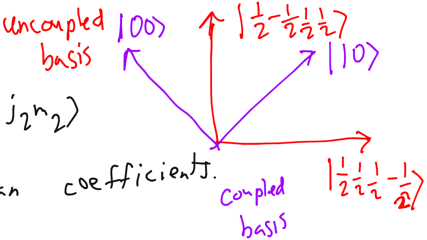
$$|1\,0\rangle = \frac{1}{\sqrt{2}}\left[|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + |\tfrac{1}{2}\,-\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\rangle\right]$$

$$|1\,-1\rangle = |\tfrac{1}{2}\,-\tfrac{1}{2}\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$|0\,0\rangle = \frac{1}{\sqrt{2}}\left[|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\,-\tfrac{1}{2}\rangle - |\tfrac{1}{2}\,-\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\rangle\right]$$

$\boxed{2}$ Some general notation:

$$|jm\rangle = \sum \underbrace{C^{j_1 j_2 j}_{m_1 m_2 m}}_{\text{Clebsch-Gordan coefficients.}} |j_1 m_1 j_2 m_2\rangle$$



Claim: $C^{j_1 j_2 j}_{m_1 m_2 m} = \underbrace{\langle j_1 m_1 j_2 m_2 | j m \rangle}$

$$\langle j_1 m_1' j_2 m_2' | \left[|jm\rangle = \sum C \, |j_1 m_1 j_2 m_2\rangle \right]$$

$$\langle j_1 m_1' j_2 m_2' | jm\rangle = \sum_{m_1 m_2} C^{j_1 j_2 j}_{m_1 m_2 m} \underbrace{\langle j_1 m_1' j_2 m_2' | j_1 m_1 j_2 m_2\rangle}_{= \begin{cases} 1 & m_1 = m_1' \ \& \ m_2 = m_2' \\ 0 & \text{otherwise} \end{cases}}$$

$$= C^{j_1 j_2 j}_{m_1' m_2' m}$$

$\boxed{3}$ Finding C–G coeff: $\frac{1}{2}\otimes\frac{1}{2} = 0\oplus 1$:

$m_1 + m_2 = m$

$\left(\frac{1}{2}+\frac{1}{2}=1\right)$

Only one state has $m=1$ $(J_z = 1\cdot\hbar)$, $\underbrace{j=1}$: $|\underset{j_1}{\tfrac{1}{2}}\ \underset{m_1}{\tfrac{1}{2}}\ \underset{j_2}{\tfrac{1}{2}}\ \underset{m_2}{\tfrac{1}{2}}\rangle = |1\,1\rangle$

$(\vec{J}_1+\vec{J}_2)^2 = \vec{J}^2 = (1+1)\cdot 1\cdot\hbar^2$

$(j+1)j\hbar^2$

$m=-1$ & $j=1$: $|1\,-1\rangle = |\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$

Find $|1\,0\rangle \propto J_-|1\,1\rangle = [J_x - iJ_y]\,|1\,1\rangle$

$= [J_{1x}+J_{2x} - i(J_{1y}+J_{2y})]\,|1\,1\rangle$

$= [\underbrace{J_{1-}} + \underbrace{J_{2-}}]\,|\underbrace{\tfrac{1}{2}\,\tfrac{1}{2}}_{j_1\ m_1}\,\underbrace{\tfrac{1}{2}\,\tfrac{1}{2}}_{m_2}\rangle$

$= \underbrace{\hbar|\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle}_{m_1\to m_1-1} + \underbrace{\hbar|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle}_{m_2\to m_2-1}$, So $|1\,0\rangle = \dfrac{|\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle + |\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle}{\sqrt{2}}$

$\boxed{4}$

$\overset{j\ m}{|0\,0\rangle}$ is an orthogonal vector to $|1\,0\rangle$

$$\frac{1}{\sqrt{2}}\left[\left|\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle\right]$$

? ?

$$|00\rangle = a\left|\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \cancel{b}^{\,-a}\left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle$$

$$\langle 10|00\rangle = \frac{1}{\sqrt{2}}\cdot a + \frac{1}{\sqrt{2}}b = 0, \quad \text{so} \quad a = -b.$$

$$\langle 00|00\rangle = 1 = |a|^2 + |-a|^2 = 2|a|^2. \qquad \text{Choose } a = \frac{1}{\sqrt{2}}$$

$$|00\rangle = \frac{1}{\sqrt{2}}\left[\left|\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle - \left|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle\right]\cdot \underbrace{e^{i\phi}}_{\text{doesn't matter}}$$

Recap: C-G coefficients:

$$\left\langle \tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\middle|00\right\rangle = 0$$

$$\left\langle \tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\middle|00\right\rangle = \pm\frac{1}{\sqrt{2}}$$

$$\left\langle \tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\middle|00\right\rangle = \mp\frac{1}{\sqrt{2}}$$

$$\left\langle \tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\middle|00\right\rangle = 0$$

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The full transformation:

$$
\begin{array}{c|cccc}
 & |11\rangle & |10\rangle & |1\,{-1}\rangle & |00\rangle \\
\hline
|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle & & & & \\
|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle & & & & \\
|\tfrac{1}{2}\,{-\tfrac{1}{2}}\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle & & & & \\
|\tfrac{1}{2}\,{-\tfrac{1}{2}}\,\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle & & & &
\end{array}
\rightarrow
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1/\sqrt{2} & 0 & \mp 1/\sqrt{2} \\
0 & -1/\sqrt{2} & 0 & \pm 1/\sqrt{2} \\
0 & 0 & -1 & 0
\end{pmatrix}
$$

Entries are C-G coefficients.

$\boxed{6}$ General strategy: for $j_1 \otimes j_2 = \ldots$

$$|\underset{j}{(j_1+j_2)}\,\underset{m}{(j_1+j_2)}\rangle = |j_1\,\underset{m_1}{j_1}\,j_2\,\underset{m_2}{j_2}\rangle$$

$$\big\downarrow J_-$$

(prop to)

$$\boxed{|\underset{j}{(j_1+j_2)}\,\underset{m}{(j_1+j_2-1)}\rangle \propto (J_{1-}+J_{2-})|j_1 j_1 j_2 j_2\rangle}$$

one orthogonal state w/ $m = j_1+j_2-1$

$$|(j_1+j_2-1)(j_1+j_2-1)\rangle$$

$$\big\downarrow J_-$$

$$\big\downarrow J_-$$

$$|(j_1+j_2)(j_1+j_2-2)\rangle \quad \xleftrightarrow{\text{always orthogonal}} \quad |(j_1+j_2-1)(j_1+j_2-2)\rangle$$

$m = j_1+j_2-2$ sector: one vector orthogonal to both.

$$|(j_1+j_2-2)(j_1+j_2-2)\rangle$$

$$\big\downarrow$$

$$\vdots$$

$$\big\downarrow J_-$$

$$|(j_1+j_2)(-j_1-j_2)\rangle = |j_1\,{-j_1}\,j_2\,{-j_2}\rangle$$

Found all states w/ $j = j_1+j_2$

one spin:

$$J_-|jm\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\,|j(m-1)\rangle$$

$\boxed{7}$

$$\overset{j_1}{4} \otimes \overset{j_2}{1} = 3 \oplus 4 \oplus 5$$

What is: $\underset{j\ m}{|5\,5\rangle} = |\underset{j_1}{4}\ \underset{m_1}{4}\ \underset{j_2}{1}\ \underset{m_2}{1}\rangle$ ✓

Next: $|54\rangle$

$$J_-|55\rangle = (J_{1-} + J_{2-})|4411\rangle$$

$$\hbar\sqrt{10}\,|54\rangle = \hbar\sqrt{8}\,|4311\rangle + \hbar\sqrt{2}\,|4410\rangle$$

$$|54\rangle = \sqrt{\tfrac{4}{5}}\,|4311\rangle + \sqrt{\tfrac{1}{5}}\,|4410\rangle$$

$|4311\rangle$ & $|4410\rangle$ only 2 states w/ $m = 4$

Jump to $|44\rangle$: $|44\rangle = \sqrt{\tfrac{1}{5}}\,|4311\rangle - \sqrt{\tfrac{4}{5}}\,|4410\rangle$

Calculate: $|43\rangle$

$$J_-|44\rangle = (J_{1-} + J_{2-})\left[\sqrt{\tfrac{1}{5}}\,|4311\rangle - \sqrt{\tfrac{4}{5}}\,|4410\rangle\right]$$

$$\hbar\sqrt{8}\,|43\rangle = \sqrt{\tfrac{1}{5}}\left[\hbar\sqrt{14}\,|4211\rangle + \hbar\sqrt{2}\,|4310\rangle\right] - \sqrt{\tfrac{4}{5}}\left[\hbar\sqrt{8}\,|4310\rangle + \hbar\sqrt{2}\,|441{-1}\rangle\right]$$

$$|43\rangle = \sqrt{\tfrac{7}{20}}\,|4211\rangle - \sqrt{\tfrac{4}{20}}\,|441{-1}\rangle - \sqrt{\tfrac{9}{20}}\,|4310\rangle.$$