PHYS 4410
Quantum Mechanics 2
Spring 2023

Lecture 2

Harmonic oscillator: algebraic solution

January 23

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Last time: complicated $V(x)$

$$V(x) = \underbrace{V(x_0)}_{\text{const.}} + \frac{1}{2}\underbrace{m\omega^2}_{= V''(x_0)}(x - x_0)^2 + \cdots$$

(x_0: loc. of min)

Convenience: - choose origin so $x_0 = 0$.
- shift energy $V(x_0) = 0$.

energy / Hamiltonian: $H = \underbrace{\frac{p^2}{2m}}_{\text{kinetic}} + \underbrace{\frac{1}{2}m\omega^2 x^2}_{\text{potential}}$. harmonic oscillator.

Goal: e-values of H.

$\boxed{2}$ Pre-processing: "remove" as many m/ω (const.) as I can.

Idea: work in "natural units"

collect constants... determine the units.

$[m] = kg = [M]$ (units of mass)

$[\omega] = s^{-1} = [T]^{-1}$

$[\hbar] = J \cdot s = [M][L]^2[T]^{-1}$

Solve for $[M],[L],[T]$:

$[M] = m$

$[T] = 1/\omega$.

$[L] = \sqrt{\frac{\hbar}{m\omega}}$

Rescale: position x:

$$\tilde{x} = \frac{x}{[L]} = \sqrt{\frac{m\omega}{\hbar}}\, x$$

momentum:

$$\tilde{p} = \frac{p}{\frac{[M][L]}{[T]}} = \frac{p}{\sqrt{\hbar m\omega}}$$

$\boxed{3}$

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$= \frac{(\sqrt{\hbar m\omega}\,\tilde{p})^2}{2m} + \frac{1}{2} m\omega^2 \left(\sqrt{\frac{\hbar}{m\omega}}\,\tilde{x}\right)^2$$

$$= \frac{\hbar\omega}{2}\left[\tilde{p}^2 + \tilde{x}^2\right]$$

[energy]

Define $\underbrace{\tilde{H} = \frac{H}{\hbar\omega} = \frac{1}{2}\left[\tilde{p}^2 + \tilde{x}^2\right]}_{\text{work w/ this.}}$. NO "const." left.

Heuristic: "work in units where $\hbar = m = \omega = 1$".

or... "measuring time in $1/\omega$, not s".

For today: drop $\sim$: $\tilde{H} \to H$. Goal: $H|\psi\rangle = \underset{\text{find!}}{E}|\psi\rangle$

$\boxed{4}$ $H = \frac{1}{2}(p^2 + x^2) \overset{\text{if classical}}{=} \frac{1}{2}(x - ip)(x + ip)$ $[i = \sqrt{-1}]$.

$[p = -i\, d/dx]$ ~~if classical~~ quantum...

$$(x-ip)(x+ip) = xx - ipx + ixp + (-i^2)p^2 = x^2 + p^2 + \underbrace{i\,[x,p]}_{-1}$$

$[x,p] = xp - px$

Claim: $[x,p] = i$

$$[x,p]\, f(x) = x\left(-i\frac{df}{dx}\right) - \left(-i\frac{d}{dx}(xf)\right) = if, \text{ so } [x,p] = i.$$

Therefore: $H = \frac{1}{2}(x-ip)(x+ip) + 1/2$ $\quad \{(x-ip)(x+ip) = 2H - 1\}$

$a = \dfrac{x+ip}{\sqrt{2}}$ (lowering operator)

$\updownarrow$ Herm. conj.

$a^\dagger = \dfrac{x-ip}{\sqrt{2}}$ (raising operator)

$$[a, a^\dagger] = \left[\frac{x+ip}{\sqrt{2}}, \frac{x-ip}{\sqrt{2}}\right]$$
$$= \frac{1}{2}\left(\cancel{[x,x]} + i[p,x] + \cdots\right)$$
$$= 1$$

$\boxed{5}$ $H = a^\dagger a + \frac{1}{2}$ and $[a, a^\dagger] = 1.$
$aa^\dagger - a^\dagger a$

Suppose $H|E\rangle = E|E\rangle$.

$$H(a^\dagger|E\rangle) = (a^\dagger a + \tfrac{1}{2})a^\dagger|E\rangle = (a^\dagger a a^\dagger + \tfrac{1}{2}a^\dagger)|E\rangle$$
$$= a^\dagger(a a^\dagger + \tfrac{1}{2})|E\rangle$$
$$= a^\dagger(a^\dagger a + [a, a^\dagger] + \tfrac{1}{2})|E\rangle$$
$$= a^\dagger(H + 1)|E\rangle.$$
$$= a^\dagger(E+1)|E\rangle = (E+1)(a^\dagger|E\rangle)$$

Also: $a|E\rangle$ is eigenstate of H w/ eigenvalue $E-1$:

$$H(a|E\rangle) = (E-1)(a|E\rangle).$$

$\boxed{6}$ Claim: if $H|E\rangle = E|E\rangle$, then $E \geq 0$.

$\langle E|H|E\rangle = E$ [if $|E\rangle$ normalized].

$= \langle E|\frac{x^2+p^2}{2}|E\rangle \geq 0.$ $\quad \langle E|x^2|E\rangle = \int_{-\infty}^{\infty} dx\; x^2 |\psi|^2 \geq 0.$

$a^{\text{"large"}}|E\rangle$ is in e-state w/ $E' < 0$?

$= E - \text{"large"}.$

Resolution: must be state for which $\underline{a|\psi_0\rangle = 0}$ (null vector)

$$\langle\psi_0|H|\psi_0\rangle = \underbrace{\langle\psi_0|a^\dagger a|\psi_0\rangle}_{(a|\psi_0\rangle)^\dagger(a|\psi_0\rangle)=0} + \frac{1}{2}\underbrace{\langle\psi_0|\psi_0\rangle}_{=1} = 1/2.$$

"natural units"

So g.s. energy of H.O. $E_0 = 1/2 = \frac{1}{2}\hbar\omega$

Act w/ $a^\dagger$: $(a^\dagger)^n|\psi_0\rangle$ is e-state of H w/ energy $E_n = \hbar\omega(n+\frac{1}{2})$.