PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 29

Time-dependent perturbation theory

April ~~5~~ 7

$\boxed{1}$ $$H(t) = \underbrace{H_0}_{\text{exactly solved}} + \underbrace{V(t)}_{\text{small perturbation}}$$

Goal: Given $|\psi(0)\rangle$, how does $V(t)$ modify $|\psi(t)\rangle$?

Often: start w/ $|n^{(0)}\rangle$; what's $P_{n\to m}(t)$

[prob. transition to $|m^{(0)}\rangle$].

Applications:
- qubit noise (HW10)
- lec 31, 32: interaction of atoms w/ photons

$\boxed{2}$ Zeroth order: $\underbrace{\frac{d}{dt}|\psi^{(0)}(t)\rangle = -\frac{i}{\hbar}H_0|\psi^{(0)}(t)\rangle}$

[time-dep. Schrödinger]

Solution? Suppose $|\psi^{(0)}(0)\rangle = \sum c_n|n^{(0)}\rangle = \sum c_n|n\rangle$

$$|\psi^{(0)}(t)\rangle = \sum c_n e^{-iE_nt/\hbar}|n\rangle$$

Matrix exponential:

$$|\psi^{(0)}(t)\rangle = \underbrace{e^{-iH_0t/\hbar}}|\psi^{(0)}(0)\rangle \qquad e^A = \sum_{j=0}^{\infty}\frac{1}{j!}A^j$$

$$\frac{d}{dt}|\psi^{(0)}(t)\rangle = -\frac{iH_0}{\hbar}e^{-iH_0t/\hbar}|\psi^{(0)}(0)\rangle.$$

Since any function of H_0 is diag. in H_0's eigenbasis

$$f(H_0) = \sum f(E_n)|n\rangle\langle n| \qquad e^{-iH_0t/\hbar} = \sum_n e^{-iE_nt/\hbar}|n\rangle\langle n|$$

$\boxed{3}$ For perturbation theory:

$$|\psi(t)\rangle = \sum_n \underbrace{\underbrace{c_n(t)}\, e^{-iE_n t/\hbar}}\, |n\rangle = |\psi^{(0)}(t)\rangle + \lambda|\psi^{(1)}(t)\rangle + \cdots$$

$$c_n(t) = \langle n|\psi(t)\rangle = c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} + \cdots$$

Previous calculation: $c_n^{(0)}$ is constant.

$$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}\left(H_0 + \lambda V(t)\right)|\psi(t)\rangle = -\frac{i}{\hbar}(H_0 + \lambda V)\left(|\psi^{(0)}\rangle + \lambda|\psi^{(1)}\rangle + \cdots\right)$$

Collect terms at first order in λ:

$$\frac{d}{dt}|\psi^{(1)}(t)\rangle = -\frac{i}{\hbar}\left[V|\psi^{(0)}\rangle + H_0|\psi^{(1)}\rangle\right]$$

Project onto $\langle n|$

$$\langle n|\frac{d}{dt}|\psi^{(1)}(t)\rangle$$

$$= \frac{d}{dt}\left(c_n^{(1)} e^{-iE_n t/\hbar}\right) = -\frac{i}{\hbar}\left[\langle n|V|\psi^{(0)}(t)\rangle + \overbrace{\langle n|H_0|\psi^{(1)}(t)\rangle}^{\langle n|E_n}\right]$$

$$= \left(\frac{dc_n^{(1)}}{dt} - \frac{iE_n}{\hbar}c_n^{(1)}\right)e^{-iE_n t/\hbar} = -\frac{i}{\hbar}\left[\langle n|V|\psi^{(0)}(t)\rangle + E_n c_n^{(1)} e^{-iE_n t/\hbar}\right]$$

$\boxed{4}$ $= \left(\frac{dc_n^{(1)}}{dt} - \cancel{\frac{iE_n}{\hbar} c_n^{(1)}} \right) e^{-iE_nt/\hbar} = -\frac{i}{\hbar}\left[\langle n|V|\psi^{(0)}(t)\rangle + \cancel{E_n c_n^{(1)} e^{-iE_nt/\hbar}} \right]$

$$\frac{dc_n^{(1)}}{dt} = -\frac{i}{\hbar} e^{+iE_nt/\hbar} \langle n|V|\psi^{(0)}(t)\rangle$$

$$= -\frac{i}{\hbar} \sum_m \langle n|V(t)|m\rangle e^{iE_nt/\hbar} \left(\underline{c_m^{(0)}} e^{-iE_mt/\hbar} \right)$$

$|\psi^{(0)}(0)\rangle = |i\rangle$

i = initial state

Assume that $c_n^{(0)} = \delta_{ni} = \begin{cases} 1 & n=i \\ 0 & n\neq i \end{cases}$.

Then: $\frac{dc_n^{(1)}}{dt} = -\frac{i}{\hbar} \langle n|V(t)|i\rangle e^{i(E_n-E_i)t/\hbar}$.

$$c_n^{(1)}(t) = \int_0^t ds \left(-\frac{i}{\hbar} \langle n|V(t)|i\rangle e^{i\omega_{ni}t} \right), \quad \omega_{ni} = \frac{E_n - E_i}{\hbar}$$

$\boxed{5}$ Suppose $H_0 = p^2/2m + \frac{1}{2}m\omega^2 x^2$; $V(t) = g x e^{-t/\tau}$

Initial state $|i\rangle = |0\rangle$.

$$\omega_{10} = \frac{\frac{3}{2}\hbar\omega - \frac{1}{2}\hbar\omega}{\hbar}$$

For what n does $c_n^{(1)}(t) \neq 0$?

- $c_n^{(1)}(t) \neq 0$ only if $\langle n|x|i\rangle = \langle n|x|0\rangle \neq 0$.

$$\omega_{10} = \omega$$

$$\langle n|x|0\rangle = \langle n| \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)|0\rangle = \sqrt{\frac{\hbar}{2m\omega}} \underbrace{\langle n|1\rangle}_{\delta_{n1}}$$

$$\underline{c_1^{(1)}(t)} = -\frac{i}{\hbar}\int_0^t ds \sqrt{\frac{\hbar}{2m\omega}}\, g e^{-s/\tau} e^{i\omega s} = -\frac{ig}{\sqrt{2\hbar m\omega}} \frac{\cancel{e^{(i\omega - 1/\tau)t}} - 1}{i\omega - 1/\tau}$$

At $t\to\infty$, probability of transition $P_{0\to1}$?

$$P_{0\to1} = |\langle 1|\psi(\infty)\rangle|^2 \approx |c_1^{(1)}(\infty)|^2 = \frac{g^2}{2\hbar m\omega}\frac{1}{\omega^2 + (1/\tau)^2}$$

At $\tau = \infty$, $P_{0\to1}$ still finite! [g.s. of H_0 $\approx$ g.s of $H_0 + V$].

Aside:

$$c_i^{(1)}(t) = -\frac{i}{\hbar}\int_0^t ds \underbrace{\langle i|V(t)|i\rangle}_{\text{real}} = \text{imaginary number.}$$

$$c_i^{(0)} = 1 = \text{real}$$

$$|c_i^{(0)} + c_i^{(1)}|^2 \approx 1 + \mathcal{O}(V^2)$$