PHYS 4410
Quantum Mechanics 2
Spring 2023

Lecture 3

Harmonic oscillator: problem solving

January 25

$\boxed{1}$ Review: quantum harmonic osc.

$$a = \frac{1}{\sqrt{2}}(\tilde{x} + i\tilde{p})$$

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

$$a^\dagger = \frac{1}{\sqrt{2}}(\tilde{x} - i\tilde{p})$$

$$\frac{H}{\hbar\omega} = a^\dagger a + \frac{1}{2} \leftarrow \hbar\omega \left[a^\dagger a + \frac{1}{2}\right]$$

$$[a, a^\dagger] = 1.$$

$$a^\dagger a |n\rangle = n |n\rangle$$

Spectrum of H: $E_n = \hbar\omega(n + 1/2)$ $n = 0, 1, 2, 3, \ldots$

Let $|0\rangle$ (for $n=0$) denote ground state. Normalized $\langle 0 | 0 \rangle = 1$.

$a^\dagger$ is raising: $\overline{|n\rangle} \stackrel{?}{=} (a^\dagger)^n |0\rangle$; Monday!

nth excited

$$H\,\overline{|n\rangle} = E_n \overline{|n\rangle}$$

$$\langle \bar{n} | \bar{n} \rangle = \left(a^\dagger \overline{|n-1\rangle}\right)^\dagger \left(a^\dagger \overline{|n-1\rangle}\right) = \langle \overline{n-1} | a a^\dagger | \overline{n-1} \rangle$$

$$= \langle \overline{n-1} | a^\dagger a + 1 | \overline{n-1} \rangle$$

$$= \langle \overline{n-1} | \left[\frac{H}{\hbar\omega} - \frac{1}{2}\right] + 1 | \overline{n-1} \rangle = (\sqrt{n})^2$$

$$= [(n-1) + 1] \langle \overline{n-1} | \overline{n-1} \rangle$$

$$a^\dagger |n-1\rangle = \sqrt{n}\,|n\rangle$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$\boxed{2}$ What's the groundstate $\psi_0(x)$? $\{\psi_0(x) = \langle x|0\rangle \ldots\}$

Idea: $a|0\rangle = 0 \Rightarrow$ differential equation for ψ_0?

$$a = \frac{1}{\sqrt{2}}(\tilde{x} + i\tilde{p}) = \frac{1}{\sqrt{2}}\left(\tilde{x} + i\left(-i\frac{d}{d\tilde{x}}\right)\right)$$

$$= \frac{1}{\sqrt{2}}\left(\tilde{x} + \frac{d}{d\tilde{x}}\right)$$

restore units?

$$\left(\tilde{x} + \frac{d}{d\tilde{x}}\right)\psi_0(\tilde{x}) = 0.$$

Solved by $\psi_0 = e^{-\tilde{x}^2/2} \cdot \cancel{\text{const.}}\ \pi^{-1/4}$

$$\tilde{x} = \frac{x}{\underbrace{\sqrt{\hbar/m\omega}}}$$

$$\psi_0(x) = \# \cdot e^{-x^2 m\omega/2\hbar}$$

$$\frac{d\psi_0}{dx} = -x\psi_0$$

$$\frac{d\psi_0}{\psi_0} = -x\,dx$$

$$\ln\psi_0 = -x^2/2 + \text{const.}$$

$\boxed{3}$ What is Δx in ground state?

$$\Delta x = \sqrt{\langle 0|x^2|0\rangle - \langle 0|x|0\rangle^2}$$

To calculate: $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^\dagger)$

$(a|0\rangle)^\dagger|0\rangle$

$$\langle 0|x|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\underbrace{\langle 0|a+a^\dagger|0\rangle}$$

$$= \cancel{\langle 0|a|0\rangle} + \langle 0|a^\dagger|0\rangle = \langle 0|1\rangle = 0.$$

$$\langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega}\,\underbrace{\langle 0|(a+a^\dagger)(a+a^\dagger)|0\rangle}$$

$$\langle 0|\,\cancel{aa} + aa^\dagger + \cancel{a^\dagger a} + a^\dagger a^\dagger|0\rangle = \langle 0|aa^\dagger|0\rangle \quad \cancel{\langle 0|2\rangle}\sqrt{2}$$

$$= \langle 0|a|1\rangle = \sqrt{1}\,\langle 0|0\rangle = 1$$

$(a+a^\dagger)|0\rangle = |1\rangle.$

4 Consider state: $|\psi(0)\rangle = A\left[|0\rangle - |1\rangle\right]$.

What's A?

$A = 1/\sqrt{2}$: $\quad 1 = \langle\psi(0)|\psi(0)\rangle = |A|^2(1+1)$ or $2A^2 = 1$.

What's $|\psi(t)\rangle$?

If $|\psi(0)\rangle = \sum c_n |n\rangle$, $\quad |\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar}|n\rangle$

$\frac{E_n}{\hbar} = \omega(n+1/2)$, so

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\omega t/2}|0\rangle - e^{-3i\omega t/2}|1\rangle\right]$$

$$= \frac{e^{-i\omega t/2}}{\sqrt{2}}\left[|0\rangle - e^{-i\omega t}|1\rangle\right].$$

$\boxed{5}$ What's $\langle \psi(t)|x|\psi(t)\rangle$?

$$\langle \psi(t)|x|\psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle\psi(t)|\underbrace{a+a^\dagger|\psi(t)\rangle}\right]$$

$$\frac{1}{\sqrt{2}}\left[(a+a^\dagger)|0\rangle - e^{-i\omega t}(a+a^\dagger)|1\rangle\right]$$

$$|1\rangle \quad - e^{-i\omega t}\left(\sqrt{1}\,|0\rangle + \cancel{\sqrt{2}\,|2\rangle}\right)$$

$$\sqrt{\frac{\hbar}{2m\omega}}\left[\frac{\langle 0| - e^{i\omega t}\langle 1|}{\sqrt{2}}\right]\left[\frac{|1\rangle - e^{-i\omega t}|0\rangle}{\sqrt{2}}\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[-e^{i\omega t}\langle 1|1\rangle - \langle 0|0\rangle e^{-i\omega t}\right]\frac{1}{2}$$

$$= -\cos(\omega t)\sqrt{\frac{\hbar}{2m\omega}}$$