PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 30

Fermi's golden rule

April 10

$\boxed{1}$ Last time: $H(t) = \underbrace{H_0}_{\text{exactly solv.}} + \lambda \underbrace{V(t)}_{\text{small pert.}}$ $[H'(t)]$

[not $|n^{(0)}\rangle$]

$\downarrow$

$H_0 |n\rangle = E_n |n\rangle$.

$|\psi(t)\rangle = \underbrace{|\psi^{(0)}(t)\rangle}$ $+ \lambda |\psi^{(1)}(t)\rangle + \cdots$

$i =$ "initial state"

$|\psi(t)\rangle = \sum [c_n^{(0)} + \lambda c_n^{(1)}(t) + \cdots] e^{-iE_n t/\hbar} |n\rangle$, $c_n^{(0)} = \begin{cases} 1 & n = i \\ 0 & n \neq i \end{cases}$

What's probability transition from $|i\rangle \to |f\rangle$ (final):

$$c_f^{(1)}(t) = -\frac{i}{\hbar} \int_0^t ds \, \langle f | V(s) | i \rangle e^{i\omega_{fi} s}, \quad \omega_{fi} = \frac{E_f - E_i}{\hbar}$$

$$\boxed{P_{i \to f} = |c_f^{(1)}(t)|^2}$$

$\boxed{2}$ Today: $V(t) = \underbrace{V}_{\text{time-independent}} \cdot 2\cos(\omega t) = V \cdot \underbrace{\left[e^{i\omega t} + e^{-i\omega t}\right]}$

$$c_f^{(1)}(t) = -\frac{i}{\hbar}\int_0^t ds\ \langle f|V|i\rangle \left(e^{i\omega s} + e^{-i\omega s}\right) e^{+i\omega_{fi} s}$$

$$\left[\int dx\, e^{ax} = \frac{e^{ax}}{a}\right]$$

$$= -\frac{i}{\hbar}\langle f|V|i\rangle \left[\cancel{\frac{e^{i(\omega_{fi}+\omega)t} - 1}{i(\omega_{fi}+\omega)}} + \underbrace{\frac{e^{i(\omega_{fi}-\omega)t} - 1}{i(\omega_{fi}-\omega)}}\right]$$

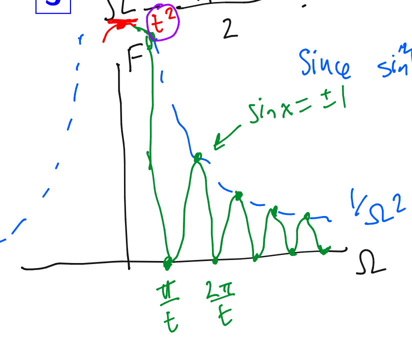
Resonance: $\omega_{fi} > 0$, $\omega > 0$ large $\omega \to \omega_{fi}$

$$c_f^{(1)}(t) = -\frac{\langle f|V|i\rangle}{(\omega_{fi}-\omega)\hbar}\underbrace{\left[e^{i\frac{t}{2}(\omega_{fi}-\omega)} - e^{-i t/2(\omega_{fi}-\omega)}\right]}_{2i\sin\left(\frac{t}{2}(\omega_{fi}-\omega)\right)}\underbrace{e^{i t/2(\omega_{fi}-\omega)}}_{|\cdots| = 1}$$

$$P_{i\to f}(t) = |c_f^{(1)}(t)|^2 = \frac{|\langle f|V|i\rangle|^2}{\hbar^2(\omega_{fi}-\omega)^2}\cdot 4\sin^2\left(\frac{t}{2}(\omega_{fi}-\omega)\right)$$

$\boxed{3}$ $\Omega = \frac{\omega_{fi} - \omega}{2}$: $P_{i\to f} \sim F(\Omega)$ $F = \frac{\sin^2(\Omega t)}{\Omega^2}$

Since $\sin^2(\Omega t) \leq 1$



$\sin^2 x = 0$

if $x = 0, \pi, 2\pi, \ldots$

$F(\Omega \to 0)$?

$\approx \frac{(\Omega t)^2}{\Omega^2} = t^2$

Claim:

$R_{i\to f} = \frac{P_{i\to f}}{t}$

rate of transition / unit time

$\underline{R_{i\to f} \sim \delta(\Omega)}$ [t-independent R]

Review: Dirac δ:

$$\int_{-\infty}^{\infty} d\Omega \, \delta(\Omega) \, h(\Omega) = h(0)$$

$$\delta(\Omega) = \begin{cases} \infty & \Omega = 0 \\ 0 & \Omega \neq 0 \end{cases} .$$

$$\frac{P_{i\to f}}{t} = R_{i\to f} \sim \frac{\sin^2(\Omega t)}{\Omega^2 t} .$$

If $\Omega \neq 0$, $t \to \infty$:
$R \to 0$.

Ask: $$\int d\Omega \, h(\Omega) \, R_{i\to f}(\Omega) \sim \int d\Omega \, h(\Omega) \frac{\sin^2(\Omega t)}{(\Omega t)^2} \, t$$

$z = \Omega t$: $$\sim \int dz \, h\left(\frac{z}{t}\right) \frac{\sin^2 z}{z} \xrightarrow{t\to\infty} h(0) \underbrace{\int_{-\infty}^{\infty} dz \frac{\sin^2 z}{z^2}}_{\neq 0} .$$

Formulas are consistent if:

$$\boxed{R_{i\to f} = \frac{2\pi}{\hbar^2} \delta(\omega_{fi} - \omega) \left| \langle f | V | i \rangle \right|^2}$$

Fermi's golden rule.

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Example: $H_0 = A \begin{array}{cccc} & |1\rangle & |2\rangle & |3\rangle \\ & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \end{array}$ $\quad V = \varepsilon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \cos(\omega_0 t)$

$\langle f | V | i \rangle = 0.$

Suppose initial state is $|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

What ω_0 induce transitions $(t \to \infty)$? To what states?

No allowed transitions!

Any transitions? If $|i\rangle = |2\rangle$ or $|3\rangle \ldots$

$$|2\rangle \to |3\rangle \qquad |3\rangle \to |2\rangle$$

$$\omega_0 = \frac{A(4-2)}{\hbar} = \frac{2A}{\hbar}$$

$\boxed{6}$ Consider $H_0 = p^2/2m$. $V(x,t) = 2\cos(\omega t) \cdot g e^{-|x|/\xi}$.

If we start in $|p=0\rangle$: $R_{0\to \text{any}}$?

$$\langle \underline{p}|V|0\rangle = \int_{-\infty}^{\infty} dx\, e^{-ipx/\hbar}\, g e^{-|x|/\xi} = \frac{2g/\xi}{(1/\xi)^2 + (p/\hbar)^2}$$

$$R_{0\to\text{any}} = \int_{-\infty}^{\infty} dp\, R_{0\to p}(\omega) = \int_{-\infty}^{\infty} dp \cdot \frac{2\pi}{\hbar^2}\,\delta\left(\frac{p^2}{2m\hbar} - \omega\right) |\langle p|V|0\rangle|^2$$

$$= \frac{4\pi m}{\hbar\sqrt{2m\hbar\omega}} \left(\frac{2g\xi}{1 + 2m\omega\xi^2/\hbar^2}\right)^2.$$